Brian Bemman and David Meredith
Department of Architecture, Design, and Media Technology
Aalborg University
Rendsburggade 14, 9000 Aalborg, Denmark
{bb, dave}@create.aau.dk

Generating New Musical Works in the Style of Milton Babbitt

Abstract: Milton Babbitt is noted for composing twelve-tone and serial music that is both complex and highly constrained. He has written extensively on a variety of topics in music and his writings have had a profound and lasting impact on musical composition. In this article, we first review in detail his compositional process and the techniques he developed, focusing in particular on the all-partition array, time-point system, and equal-note-value strings used in his later works. Next, we describe our proposed procedure for automating his compositional process using these techniques. We conclude by using our procedure to automatically generate an entirely new musical work that we argue is in the style of Babbitt.

Introduction

Milton Babbitt (1916–2011) was a composer of twelve-tone and serial music whose works and theoretical writings had a profound impact on modern musical composition. Beginning in the 1950s and over the course of the next two decades, Babbitt formalized the twelve-tone system and established techniques such as the time-point system and the all-partition array (Babbitt 1955, 1960, 1961, 1962, 1973). Many of these techniques remain of interest to composition and music research today (Tanaka, Bemman, and Meredith 2016b; Bernstein 2017; Bemman and Meredith, forthcoming). Indeed, music theorists and computer scientists alike have written at length on these techniques and the highly constrained and often complex structures in Babbitt's music that result (Mead 1987, 1994; Bemman and Meredith 2015, 2016; Tanaka, Bemman, and Meredith 2016a; Bernstein 2017). In recent years, the sketches for many of his works have been made publicly available by the Library of Congress in Washington, DC (http://lccn.loc.gov/2014565648). Researchers are now able to examine Babbitt's compositional process in much greater detail than before.

In this article we describe the process Babbitt devised in composing his later works (from approximately 1980 to 2011), focusing in particular on the notions of the all-partition array, the time-point system, and the equal-note-value string. The constraints under which he composed his music at this time are strict, and his application of these techniques follows an often well-defined procedure. For this reason, we suggest that Babbitt's compositional process is inherently algorithmic in nature and that, therefore, many of these techniques can be modeled by machine. This line of reasoning follows years of research in the field of computational music analysis and generation, with similar efforts having been made, for example, to harmonize chorales in the style of Bach (Ebcioğlu 1987) and, more recently, to generate the structures found in Babbitt's own music (Bemman and Meredith 2016; Tanaka, Bemman, and Meredith 2016a,b). The primary purpose of this article, however, is simply to provide further insight into the nature of Babbitt's compositional process, notably through the creation of an algorithmic model. We should note that this model is not necessarily intended to be a tool for computer-aided algorithmic composition nor an explanation for how other twelve-tone composers (e.g., Webern, Schoenberg, Carter) might have composed their music, although such uses would be welcomed.

In the remainder of this article, we present our proposed procedure for automating Babbitt's compositional process in his later works. First, we introduce a method for generating the pitches and time points of a piece from its all-partition array. Then we show how this same method can be used to generate pitch repetitions (i.e., a single pitch that occurs at more than one point in time), which occur frequently in Babbitt's music. Next we describe a method for determining the rhythms in a piece as well as the placement of rests and ties. We conclude by using this procedure to automatically generate a novel musical work that we believe to be in the style of Babbitt.

Computer Music Journal, 42:1, pp. 60–79, Spring 2018
doi:10.1162/COMJ_a_00451

Figure 1. Excerpt containing three ordered mosaics from the all-partition array found in Babbitt's None but the Lonely Flute. *Each ordered mosaic is a partitioning of the aggregate that has been ordered from top to bottom into sets of pitch classes each ordered from left to right.*

1.	7 2		2 3 1 9 8 5
2.	6		6 11 10 0
3.	11	0 5 7	
4.	8 3 4	2 9 10	
5.	5 10 9	11 3 4	4 7
6.	0 1	1 6 8	

All-Partition Array

Both the pitch and the rhythmic content in Babbitt's later works are organized according to a structure known as the *all-partition array* (Mead 1994; Bemman and Meredith 2016). In an all-partition array, Babbitt constructed aggregates (i.e., collections of the twelve musical pitch classes) so that each was a distinct set of partitioned pitch-class segments called an *ordered mosaic* (Bemman and Meredith 2016). For example, $\langle\langle 2,0,1,3\rangle, \langle 5,4,6,7\rangle, \langle 8,9,11,10\rangle\rangle$ is one ordered mosaic made up of three pitch-class segments of length 4. We denote ordered sets using angle brackets, $\langle\cdot\rangle$, and unordered sets with braces, $\{\cdot\}$. As is standard in musical set theory, we use numbers to denote pitch classes of the equal-tempered chromatic scale independent of octave register, where C $= 0$, C♯ $= 1$, D $= 2, \ldots,$ B $= 11$. A second ordered mosaic might contain three pitch-class segments in which two are of length 5 and one is of length 2, e.g., $\langle\langle 6,1,11,4,2\rangle, \langle 5,7\rangle, \langle 0,9,8,3,10\rangle\rangle$. Each pitch-class segment in an ordered mosaic is then assigned to a "voice" on the musical surface, forming what is known as a *lyne* (Mead 1994). Collectively, these pitch-class segments in each lyne for all ordered mosaics form a concatenation of twelve-tone rows that may or may not contain repetitions. Figure 1 shows an excerpt from the all-partition array in Babbitt's *None but the Lonely Flute* with six such lynes and the first three of its ordered mosaics.

Note, in Figure 1, that the pitch-class segments in the first ordered mosaic are distributed across each of the six possible lynes but that in the third ordered mosaic, these pitch-class segments appear in only three lynes—1, 2, and 5. In the third ordered mosaic, the first pitch class in each segment is a repetition from the previous ordered mosaics: pitch classes 2 and 6 from the first ordered mosaic and pitch class 4 from the second ordered mosaic. An all-partition array must contain a number of ordered mosaics equal to the number of distinct ways in which twelve can be partitioned into k parts (i.e., lynes) or fewer, with each of these partitions represented by a distinct ordered mosaic. Babbitt used four-, six-, and twelve-part all-partition arrays containing 34, 58, and 77 ordered mosaics, respectively. In earlier publications we provide a more detailed look at the mathematical construction of an all-partition array and the difficulty associated with generating one (cf. Bemman and Meredith 2016 and Tanaka et al. 2016a,b).

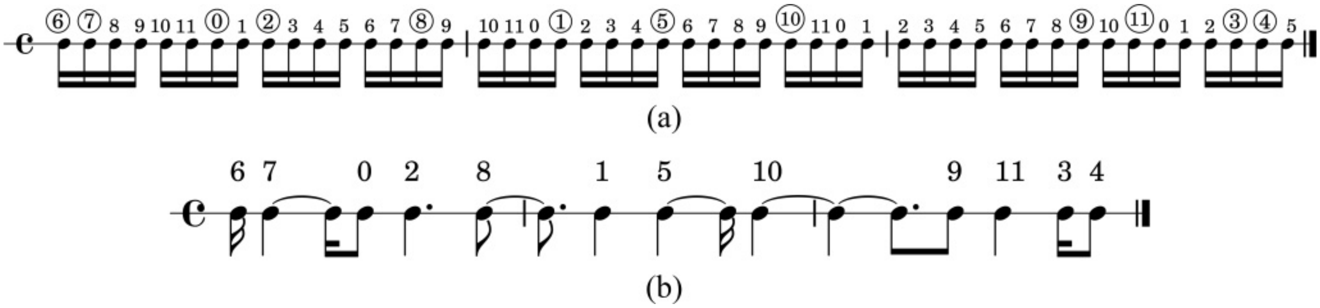
Time-Point System

Babbitt himself laid the foundations for his time-point system and outlined general principles for applying it to composition (Babbitt 1962). With the time-point system, Babbitt sought to derive a correspondence between the twelve-tone row and time. By replacing the interval of a half-step in a twelve-tone row with a fixed period of time called a *unit*, the time-point intervals (analogous to directed pitch-class intervals) between adjacent members become lengths of time measured in units rather than pitch intervals measured in semitones. In his later works, Babbitt typically used a sixteenth note as the unit (Bernstein 2017). Figure 2 shows a left-to-right linear ordering of twelve time points from the opening of Babbitt's *None but the Lonely Flute* using a sixteenth note as the unit and one possible rhythmic representation (not used by Babbitt in this piece).

Note, in Figure 2, how time points denote onsets in time corresponding to new rhythmic events. In Figure 2b, these rhythmic events have a duration equal to the interonset intervals of each time point. Babbitt often sought less straightforward rhythmic interpretations of his time points than that shown in Figure 2b, however, and in the following sections we will see in greater detail how his use of additional techniques gave his later works their characteristically varied and often complex rhythms.

Figure 2. A linear ordering of twelve time points set against a grid of sixteenth-note units (a) and one possible rhythmic interpretation of this ordering (b), in which duration is equal to interonset interval, meaning there are no rests or overlapping notes.

Figure 3. Pitch-class ordered mosaic (PcOM) distinguished by register (a) and time-point class ordered mosaic (TpcOM) distinguished by dynamic level (b).



Babbitt's Compositional Process in Later Works

In this section, we describe in detail Babbitt's compositional process as found in his later works. Many of the techniques he used in this process have been described elsewhere (Mead 1994; Bernstein 2017), yet a thorough understanding of how exactly this process comes to form the musical surface of his works is essential in explaining how we have automated this process.

$C_6 - B_6$	7	2	
	6		
$C_5 - B_5$	11		
	8	3	4
$C_4 - B_4$	5	10	9
	0	1	

(a)

ff	7	2	
f	6		
mf	11		
mp	8	3	4
p	5	10	9
pp	0	1	

(b)

Figure 3

Linear Orderings from Ordered Mosaics

In many of Babbitt's later works based on the all-partition array, the available linear orderings of pitch classes and time-point classes are the same because they are both constructed from the same ordered mosaics. We introduced the term *linear ordering* in an earlier publication (Bemman and Meredith, forthcoming), and it is equivalent to what Babbitt variously called a *secondary-set* (Babbitt 1961, p. 86) or *linear aggregate* (Babbitt 1973), and what Leong and McNutt (2005) refer to as an *aggregate realization*. A linear ordering differs from a twelve-tone row both in the way it is constructed (i.e., from an ordered mosaic and not from the four transformations of transposition, inversion, retrograde, and retrograde inversion) and the number of elements it can contain (i.e., more than twelve). Typically, Babbitt used register to distinguish pitch-class segments, and dynamic level to distinguish time-point class segments (Mead 1994). Figure 3a shows a pitch-class ordered mosaic (hereafter abbreviated "PcOM"), and Figure 3b displays a time-point class ordered mosaic (hereafter abbreviated "TpcOM"), both taken from the same all-partition array.

An ordered mosaic places constraints on the possible orderings that can be constructed from it. Each segment in a mosaic is ordered, meaning that elements, when "linearized" to form an ordering (e.g., as in Figure 2b), must remain in the left-to-right order in which they occur in their segment. For example, in Figure 3, possible orderings from both mosaics may begin with $\langle 7, 2, \ldots \rangle$ but not $\langle 2, 7, \ldots \rangle$. Similarly, elements from other segments may intervene, so long as their left-to-right order is not violated. For example, $\langle 7, 8, 2, 3 \ldots \rangle$ is allowed but not $\langle 7, 3, 2, 8 \ldots \rangle$. Figure 4a shows a possible ordering of pitch classes. Figure 4b indicates a possible ordering of time-point classes taken from the ordered mosaics in Figures 3a and 3b, respectively.

Note, in Figure 4a, how segments of pitch classes are distinguished from each other by pitch-register and, in Figure 4b, how the changes in dynamic level mark the arrival of a time point belonging to a different segment. Perceptually, these new time points act as both temporal boundaries of local

Figure 4. Opening orderings of pitch classes (a) and time-point classes (b) from Babbitt's None but the Lonely Flute. *The musical notations on the right are taken from the respective ordered mosaics on the left. Note that in in the pitch-class ordering no time information is specified, nor is any pitch information specified in the time-point ordering.*

Figure 5. Opening of Babbitt's None but the Lonely Flute *corresponding to the linear orderings of pitch classes and time-point classes shown in Figure 4. Note that the vertical dashed line in measure 5 marks the boundary between the first and second linear orderings of pitch classes.*

Figure 4

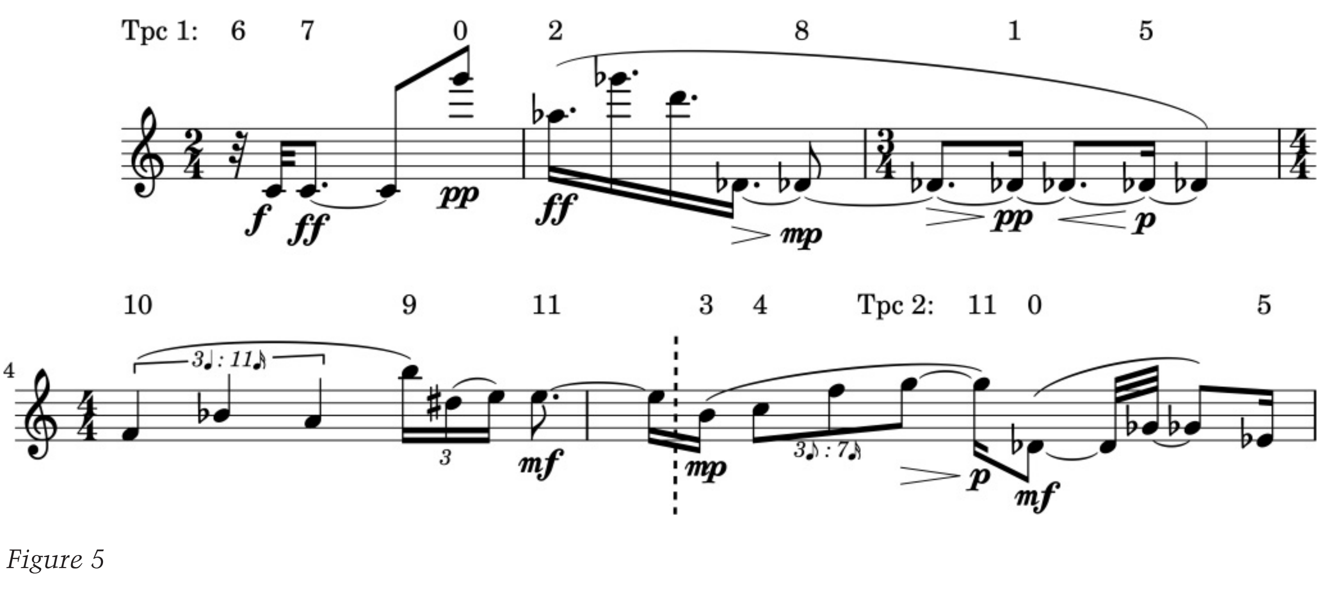
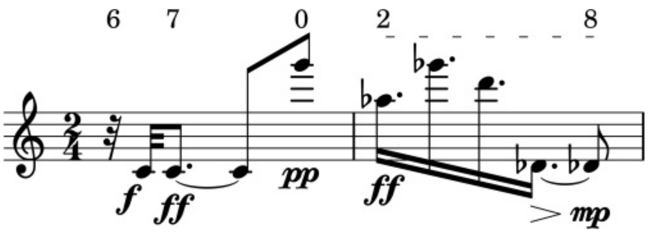


Figure 5

events and reminders of a global stream of temporal events unfolding in each time-point class segment, made clear by differing dynamic levels.

The musical surface of Babbitt's later works is formed by uniting the pitch information specified by a linear ordering constructed from a PcOM with the timing information specified by some rhythmic interpretation of a linear ordering constructed from a TpcOM. Figure 5 shows the opening of Babbitt's *None but the Lonely Flute* and how the ordering of

Figure 6. Four equal-note-value strings (one for each time-point interval) in an excerpt from the opening of Babbitt's None but the Lonely Flute *shown in Figure 5.*



pitch-classes shown in Figure 4a and the ordering of time-point classes shown in Figure 4b have been united to form the musical surface.

In Figure 5, note that on the musical surface, depending on the chosen rhythmic interpretation, pitches generally proceed faster than the time points (Mead 1994). For example, at time point 2 there are four corresponding pitch classes: 8, 6, 2, and 1. Nevertheless, Babbitt is often careful to ensure that time points are allowed to "catch up" as, for example, at time points 8, 1, and 5, where there is only a single pitch class, 1. Indeed, Mead has noted that Babbitt has had a "longtime predilection for manifesting similar sorts of distributions of events in different domains over different spans of time" (Mead 1994, p. 49). It is clear then that the chosen rhythmic interpretation of time points, as shown, for example, in Figure 4b, is crucial to maintaining such a uniform distribution of pitch and time events.

Equal-Note-Value Strings and Rhythm

A rhythmic interpretation of an ordering of time points, as shown for example in Figure 4b, is determined by Babbitt's choice of *equal-note-value strings* taken from a PcOM to form an ordering of pitch classes. An equal-note-value string is a string of *n* pitch classes that subdivide a time-point interval into *n* equal durations (i.e., note values) (Mead 1994; Bernstein 2017). Figure 6 provides an example of Babbitt's use of equal-note-value strings from the opening of *None but the Lonely Flute*.

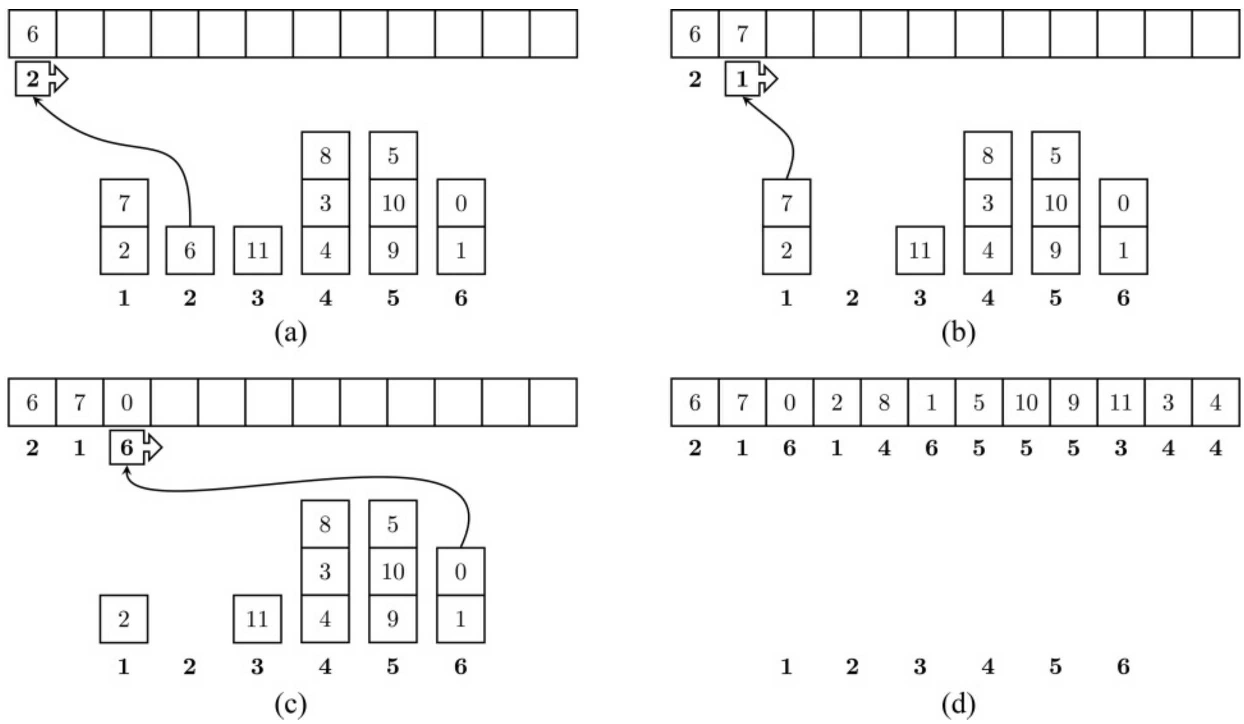
As shown by the dotted line in Figure 6, an equal-note-value string containing the pitch classes 8, 6, 2 and 1 equally subdivides the time-point interval of 6, occurring between time points 2 and 8. Because the unit in this ordering of time points is equal to a sixteenth note, this time-point interval of 6 is equal to a dotted quarter note. This interval has then been equally divided by each of its equal-note-value string's four pitch classes into four durations, each equal to a dotted sixteenth note. In fact, each time-point interval in the example shown in Figure 6 contains an equal-note-value string. For example, there is an equal-note-value string of length 1 between time points 0 and 2 (pitch-class 7) and between time points 7 and 0 (a tied note pitch-class 0). Between time points 6 and 7 there is an equal-note-value string of length 2 in which the first member is a rest. We will see in the following sections, in our discussion of automating Babbitt's compositional process, how exactly Babbitt determined the rests, ties, and repetitions of pitches that appear frequently in his music.

Generating Orderings from Ordered Mosaics

In this section, we begin the discussion of our proposed procedure for modeling Babbitt's compositional process described previously. In computer science, a *stack* is an abstract data type that stores a collection of elements using a "last-in-first-out" protocol (for a more detailed introduction, see, e.g., Cormen et al. 2009, pp. 232–236). A stack has two associated operations, *pop* and *push*, that remove its top element and insert an element at the top of the stack, respectively. By representing each segment of an ordered mosaic as a stack, such that the leftmost element lies on the top of each stack, we can account for the left-to-right order in which time-point classes or pitch classes must be taken from each segment when constructing an ordering. Figure 7 shows how an ordering of time-point classes or pitch classes can be generated from an ordered mosaic represented as a sequence of stacks.

Note, in Figure 7a–c, that at each step the top element of each indicated stack is popped and stored from left-to-right to form the sequence of elements above. As this process unfolds, the stack number is simultaneously stored below its corresponding element. In Figure 7d, this process concludes with the completed ordering of time-point classes from Figure 4b and an empty sequence of stacks.

Figure 7. The process of selecting elements from stacks, numbered 1–6, in an ordered mosaic to generate a linear ordering of pitch-classes or time-point classes. This example begins with taking the top element from stack 2 (a), with the element then removed from the stack. This is followed by taking the top element from stack 1 (b), then by an element from stack 6 (c). The process continues until eventually all stacks are empty (d). Note that the generated ordering corresponds to the time-point class ordering shown in Figure 4b.



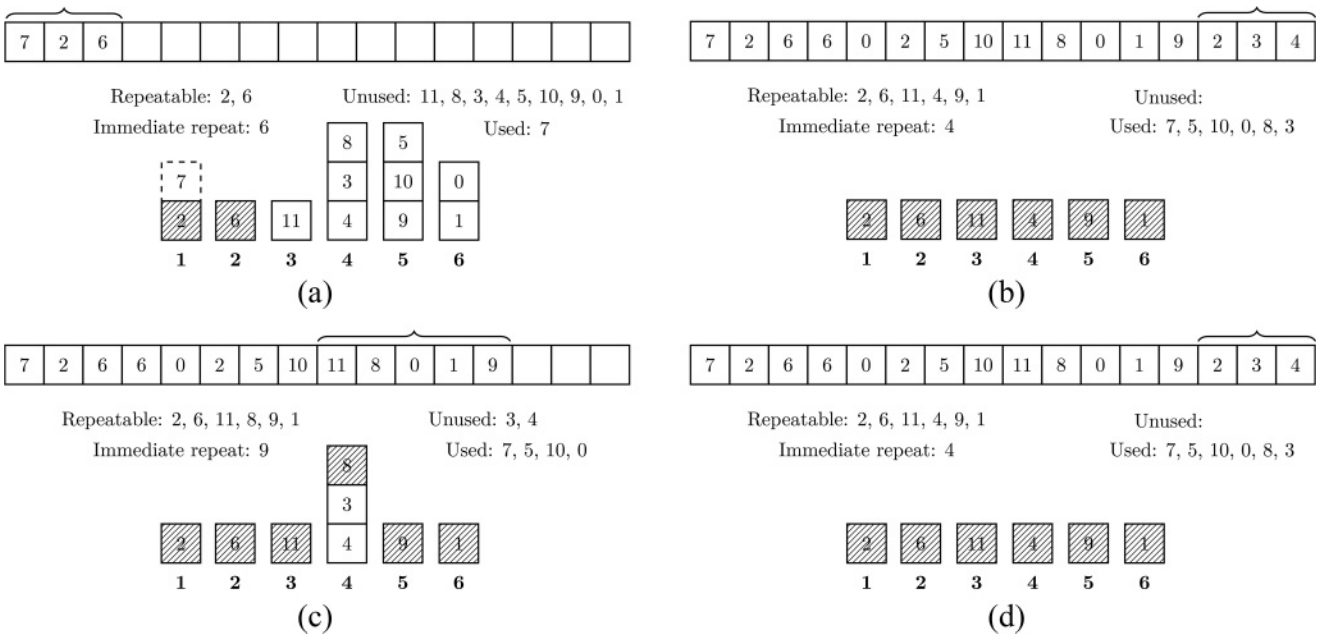
Because only the top element of each stack is a possible next choice after every chosen element, the sequence of stack numbers shown in Figure 7, ⟨2, 1, 6, 1, 4, 6, 5, 5, 5, 3, 4, 4⟩, uniquely encodes its corresponding ordering above (given the ordered mosaic). Computing all distinct permutations of such a sequence of stack numbers then corresponds to all possible orderings that can be generated from a given ordered mosaic. The ordered mosaic shown in Figure 7, for example, produces 3,324,000 distinct possible permutations. In a parallel publication we show how this number is computed using the formula for the multinomial coefficient (Bemman and Meredith, forthcoming).

Pitch-Class Repetitions in Equal-Note-Value Strings

That there are pitch repetitions in Babbitt's music distinguishes it from the works of several other twelve-tone composers. In looking at Babbitt's later works, we find two sorts of repetition, those in which a pitch is immediately repeated (as used by other twelve-tone composers), e.g., ⟨6, 6⟩, and those in which the most recently chosen pitch from a stack is repeated (largely unique to Babbitt), e.g., ⟨6, 7, 6⟩ in the ordered mosaics shown in Figure 3. When and where these repetitions occur in an ordering of pitch classes are determined, in part, by the equal-note-value strings used to construct the linear ordering. Figure 8 shows the process of constructing a linear ordering of pitch classes with repetitions, this time, by grouping pitch classes from its PcOM into equal-note-value strings.

Note, in Figure 8, that pitch classes in a PcOM can be in any one of three states in the process of constructing a pitch-class ordering: (1) unused (indicated by the white boxes), (2) repeatable (indicated by the shaded boxes), or (3) used (indicated by their removal from a stack). In Figure 8a, after the first equal-note-value string is generated, the repeatable pitch classes are 6 and 2, but not 7, as 2 and 7 belong to the same segment (i.e., stack) and 7 is not the most recent pitch class to be taken from this segment. In Figure 8b, this equal-note-value string contains two repeatable pitch classes, 6 and 2,

Figure 8. Process of generating a pitch-class ordering of equal-note-value strings containing repetitions from stacks, numbered 1–6, in a PcOM. In each step, one equal-note-value string (indicated by a brace above the ordering at top) is constructed for a given time-point class. Shaded boxes below the ordering indicate pitch classes that are repeatable and white boxes indicate unused pitch classes.



where 6 is an immediate repetition (from Figure 8a). In Figure 8c, this equal-note-value string contains only a single repeatable pitch class, 0, that is not immediate. In Figure 8d, we have a completed linear ordering of twelve distinct pitch classes containing four equal-note-value strings for a total of 16 elements due to its four repetitions.

When and Where Pitch Repetitions Occur

The problem of determining where exactly pitch repetitions may occur in Babbitt's music is addressed by Babbitt himself, who states that "pitch repetition is not a pitch procedure, but a temporal procedure, independent of the considerations of the pitch system, and, if a time-point system is assumed, the temporal placements of such pitch repetitions are determined by the time-point structure, not by pitch considerations" (Babbitt 1962, p. 65). It is likely that Babbitt used repetitions to articulate any number of temporal or nontemporal events, including beat, meter, and syncopation, among others. In our model, however, immediate repetitions are those that occur predominantly on the beat, whereas all other repeatable pitches primarily occur off the beat.

Let us suppose in a piece we have a sequence of time points $P = \langle p_1, p_2, \ldots, p_n \rangle$. The time-point intervals of n time points form the sequence $T = \langle t_1, t_2, \ldots, t_{n-1} \rangle$, where $t_i = p_{i+1} - p_i \pmod{12}$. Whether or not a time point p_i falls on the beat in an implied meter is determined by the *prefix sum* of time-point interval t_{i-1} modulo the number of time-point units u between consecutive beats. (The *prefix sum* of a sequence of numbers is equivalent to the cumulative sum or additive sum of its elements.) We therefore define an on-the-beat indicator o_i to be 1 if and only if p_i falls on a beat, as follows:

$$o_i = \begin{cases} 1, & \text{if } \sum_{j=1}^{i-1} t_j \equiv 0 \pmod{u}; \text{ and} \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

In Equation 1, when $o_i = 1$, the equal-note-value string at time point p_i for time-point interval t_{i-1} falls on the beat. For example, in a common-time meter with four sixteenth-note units to the beat and where a time-point interval t_{i-1} has a prefix sum equal to $\sum_{j=1}^{i-1} t_j = 4$, the modulo $u = 4$ operation

Table 1. Probabilities of Pitch-Class Events

		Immediate Repetition	*Other Repetition*	*New Pitch Class*
$o_i =$	1	50%	25%	25%
	0	0%	25%	75%

Probabilities in our model of different classes of events occurring either on the beat, $o_i = 1$, or off the beat, $o_i = 0$.

would result in 0, and thus the time-point p_i falls on the beat.

As shown in Figure 8, possible pitch-class events in an equal note-value string are (1) an immediate repetition, (2) some other repetition, and (3) a new pitch class, respectively denoted ir, or, and np. To promote a variety of musically interesting events in the music generated by our model, we assign varying probabilities to these possible events at each given time-point interval. We have chosen these probabilities based on observations of the approximate number of times these musical events occur in Babbitt's music. We should note, however, that no systematic corpus analysis was used in determining this number, as very few encodings of Babbitt's works exist. Therefore, the actual number of occurrences of various events in Babbitt's music remains unknown. Nonetheless, the probabilities we arrived at are summarized in Table 1.

In the case that the chosen pitch-class event is either some other repetition or a new pitch class and there is more than one pitch class available in an ordered mosaic, we ensure that each available pitch class has an equal probability of being chosen. This occurs, for example, in Figure 8b with three other repetitions (2, 6, 0) and six unused pitch classes (11, 8, 3, 4, 9, 1). Naturally, there is only ever one available repetition that can be immediate. In Babbitt's later works, all the pitch classes in a given equal-note-value string must be distinct, implying that only the first pitch class in such a string can be an immediate repeat. The remaining pitch classes in such a string can be either other repeats or new pitch classes. In general, however, the equal-note-value strings Babbitt constructed contain in their entirety only two repetitions. We suspect this ensured for Babbitt that the lengths of pitch-class orderings on the musical surface do not grow exceedingly long with repetitions. Accordingly, the equal-note-value strings generated by our model are constrained to contain only two repetitions.

Maximum Length of an Equal-Note-Value String

As illustrated by the opening of Babbitt's *None but the Lonely Flute*, shown in Figure 5, the juxtaposition of different length equal-note-value strings and time-point intervals contributes to a musically interesting result in which the distributions of time points and pitch classes can remain approximately uniform. It seems, however, that Babbitt did not consider all combinations of string length and time-point intervals to be musically meaningful and, indeed, avoided certain combinations altogether. Consider, for example, how difficult it might be for a human to perform a twelve-note tuplet in the time of a sixteenth note at even a moderate tempo. To avoid such problems, for any given time-point interval we constrain the maximum length of its corresponding equal-note-value string.

The maximum allowable length for a given equal-note-value string can neither exceed the number of unused pitch classes in its PcOM at any given point in constructing a pitch-class ordering (i.e., the white boxes shown in the process described in Figure 8) nor can it result in subdivisions with a duration smaller than some fixed note value. The maximum length of an equal-note-value string s_i for a time-point interval t_i is given by

$$|s_i|_{max} = \min(r, t_i d), \tag{2}$$

where d is the length of the time-point unit in thirty-second notes and r is the number of unused pitch classes in the PcOM before constructing s_i. For example, when the unit in an ordering of time points is equal to a sixteenth note (i.e., $d = 2$), and the current time-point interval is 3, then an equal-note-value string of length $3 \cdot 2 = 6$ is acceptable in a PcOM with six or more unused pitch classes. Each of the durations in this equal-note-value string would then be equal to a thirty-second note.

Figure 9. An equal-note-value string, ⟨6, 11⟩, constructed from two segments in the first PcOM that references a single segment, ⟨6, 11, 10, 0⟩, belonging to the third PcOM.

1.	7	2					2	3	1	9	8	5
2.	6						6	11	10	0		
3.	11			0	5	7						
4.	8	3	4	2	9	10						
5.	5	10	9	11	3	4	4	7				
6.	0	1		1	6	8						

Generating Rests and Ties in Equal-Note-Value Strings

As illustrated in Figure 1, pitch classes in a lyne and, by extension, their corresponding segments from all PcOMs in an all-partition array are ordered as a result of the twelve-tone rows to which they belong. In Figure 8, however, we saw how an equal-note-value string generated from a PcOM can contain pitch classes not necessarily belonging to a single segment or lyne. It is therefore possible that such strings may contain an ordering of pitch classes not found in any of the twelve-tone rows in the lynes of an all-partition array. Generally speaking, Babbitt found this undesirable because it violates an essential principle of twelve-tone composition—namely, that pitch classes from a twelve-tone row appear in their given order. In Babbitt's later works, he ensured that the ordering of pitch classes in an equal-note-value string corresponded to the orderings found in these twelve-tone rows by checking that either the string in its entirety or its substrings belong to one or more segments from other PcOMs in its all-partition array. In this way, Babbitt was able to use equal-note-value strings to create a dense network of motivic ideas across a piece by linking different ordered mosaics in an all-partition array (Mead 1994; Bernstein 2017). This process of constructing equal-note-value strings that can also be constructed from other PcOMs has been called *referenced array segments* (Bernstein 2017).

The *array segments* of an all-partition array are all the pitch-class segments found in any of the PcOMs in an all-partition array. For example, the excerpt from Figure 1 contains one pitch-class segment of length 6, one segment of length 4, six segments of length 3, three segments of length 2, and two segments of length 1. A complete all-partition array, containing 34, 58, or 77 ordered mosaics, will have many more segments. An equal-note-value string s is said to *reference* an array segment a (typically in another PcOM) if a substring of s is a substring of a. For example, Figure 9 shows the excerpt from the all-partition array shown in Figure 1 with an equal-note-value string constructed from two segments in its first PcOM that references a segment in its third PcOM.

Note that in Figure 9, how the referenced array segment ⟨6, 11, 10, 0⟩ in the third PcOM contains a substring equal to the equal-note-value string ⟨6, 11⟩, constructed from the first PcOM. Moreover, in both PcOMs, the process of selecting elements (shown in Figure 7) to form this substring is not violated.

When constructing an equal-note-value string in this way, we argue that Babbitt sought to minimize the number of referenced array segments required to account for all its pitch classes. More formally, we propose that Babbitt desired a minimum cardinality set C of substrings from an equal-note-value string s (1) that covers s and (2) whose members c_i are substrings of array segments other than those used to construct s. If we return again to Figure 9, the 6 in our equal-note-value string ⟨6, 11⟩ could have referred to the segment ⟨6, 11, 10, 0⟩ in the third PcOM and the 11 could have referred to the segment ⟨11, 3, 4⟩, in the second. This would require a set $C = \langle\langle 6\rangle, \langle 11\rangle\rangle$ having a cardinality of 2, however, which would be considered less optimal than our original reference to ⟨6, 11, 10, 0⟩.

Rests and Ties

It is possible that many equal-note-value strings might not have a minimum cardinality set. It is also possible that a referenced array segment might contain more pitch classes than its equal-note-value string (as shown in Figure 9), whether it has a minimum cardinality set or not. It is in these cases that rests and ties arise. As Bernstein (2017) has noted, Babbitt indicated in his sketches the ordinal positions of a string's pitch classes in their referenced array segments. Because each reference must be a substring, numbers belonging to the same array segment must be sequential in ascending order. For example, an equal-note-value string $\langle 6, 11, 7\rangle$ would have the ordinal positions 1, 2, 3 in a reference to a single array segment $\langle\langle 6, 11, 7\rangle\rangle$ or 1, 2, 1 in a reference to two array segments, e.g., $\langle\langle 6, 11\rangle, \langle 7, 9, 2, 10\rangle\rangle$. Babbitt indicated the lengths of each referenced array segment by noting their final ordinal positions with either an underscore or parentheses. In the example using two array segments just provided, these lengths could be indicated by the following: 1,2,(1).

As Bernstein continues, an underscore indicates that this final pitch class of an array segment should "sound" on the musical surface and parentheses indicate that this pitch class should not. We take this to mean that parentheses indicate rests and that underscores, in general, indicate ties. We have observed that Babbitt does not typically embed rests in the middle of an equal-note-value string in his later works, opting instead to append or prepend them to a string. On the other hand, although ties do appear at the ends of equal-note-value strings, we only permit them to appear inside a string in our model. We have chosen to do this because it was not always possible to ensure that the pitch required to complete a tied note from the final position of one equal-note-value string was available in the first position of the next equal-note-value string. This could be due to, for example, being in a new PcOM where the required pitch does not appear as the first element in the appropriate segment or falling on an off-beat where immediate repetitions are not allowed according to Table 1. An equal-note-value string, $\langle 6, 11\rangle$, in our model, for example, that references the array segment, $\langle 6, 11, 2\rangle$, would contain a rest in its final position. This then transforms the original string of length 2 into one of length 3, $\langle 6, 11, \text{rest}\rangle$. On the other hand, a tie in our model can only occur from a referenced array segment containing more pitch classes than the substring from its equal-note-value string. This is because, for there to be a tie or "underscore," there must be a right-most pitch class in the referenced array segment that does not belong to the substring of the equal-note-value string. For example, an equal-note-value string $\langle 6, 11\rangle$ that references the two array segments, $\langle 6, 7\rangle, \langle 11\rangle$ would contain a tie in its second position, $\langle 6, \text{tie}, 11\rangle$. Table 2 shows various referenced array segments and the musical output for a given input of a string $\langle 6, 7, 0\rangle$ and a time-point interval equal to three sixteenth notes.

Note, in Table 2, how the minimum cardinality set $\langle\langle 6, 7, 0\rangle\rangle$ for the referenced array segment $\langle 6, 7, 0\rangle$ in the first row is optimal and so its corresponding musical output contains no rests or ties. In the second row, its referenced array segment, although forming a minimum cardinality set, nonetheless contains more pitch classes than its equal-note-value string. As such, the corresponding output contains a single rest at the end and we consider this slightly less optimal. In the sixth row, note how the presence of pitch-class 4 in the referenced array segment causes a tie to appear in the output. We believe this is least optimal. Finally, note that the reference in the eighth row is not possible, as it does not contain substrings that belong to this equal-note-value string (i.e., 6 and 7 are out of order). In our model, the covers $\langle\langle 6, 7\rangle, \langle 0\rangle\rangle$ and $\langle\langle 6\rangle, \langle 7, 0\rangle\rangle$ would be considered equally good.

It is important to note that equal-note-value strings in our model differ slightly from those in Babbitt's practice. In Babbitt's later works, he ensured that the referenced array segments in a piece form an exhaustive *cover* of all array segments found in its all-partition array, with every pitch class in any one segment referenced at least once. Finding such a cover of an all-partition array by equal-note-value strings is a difficult problem. Presently, we do not have a method for solving this problem. For this reason, we have adopted a "greedy" approach in which equal-note-value strings are constructed according to the first possible reference (searching from shortest-length segments first) and where a single array segment may be referenced more than once without having all array segments referenced.

Automating the Compositional Process Found in Babbitt's Later Works

We begin with an all-partition array as input, generated using either of the methods described by Bemman and Meredith (2016) or by Tanaka, Bemman, and Meredith (2016a). For each of the n ordered mosaics in this array, we first generate a

Table 2. Possible Output from One Equal-Note-Value String

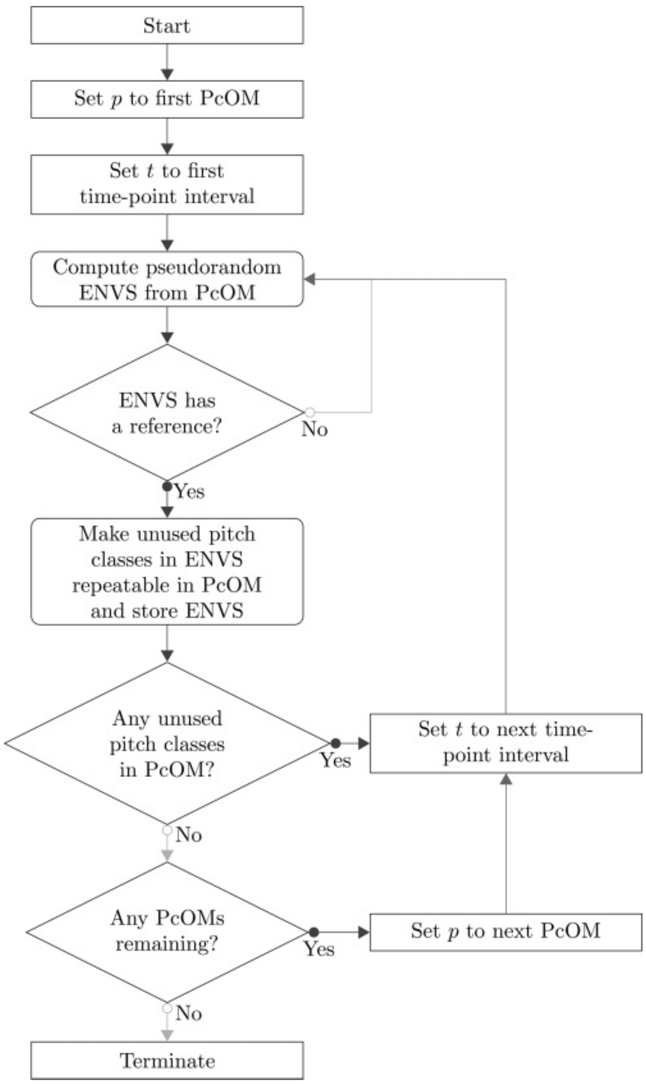
	Set of Substrings	*Referenced Array Segments*	*Output*
	$\langle\langle 6,7,0\rangle\rangle$	$\langle 6,7,0\rangle$	
↑	$\langle\langle 6,7,0\rangle\rangle$	$\langle 6,7,0,9\rangle$	
	$\langle\langle 6,7\rangle,\langle 0\rangle\rangle$	$\langle 6,7\rangle$ and $\langle 0\rangle$	
optimal covering	$\langle\langle 6,7\rangle,\langle 0\rangle\rangle$	$\langle 6,7\rangle$ and $\langle 0,9\rangle$	
	$\langle\langle 6,7\rangle,\langle 0\rangle\rangle$	$\langle 2,6,7\rangle$ and $\langle 0,9\rangle$	
	$\langle\langle 6,7\rangle,\langle 0\rangle\rangle$	$\langle 6,7,4\rangle$ and $\langle 0,9\rangle$	
	$\langle\langle 6\rangle,\langle 7\rangle,\langle 0\rangle\rangle$	$\langle 6,8\rangle$ and $\langle 7,4\rangle$ and $\langle 0,9\rangle$	
	$\langle\langle 6,7,0\rangle\rangle$	$\langle 4,9,7,6,0\rangle$	N/A

Various referenced array segments and the musical output for a given input equal-note-value string s $= \langle 6,7,0\rangle$ *and time-point interval equal to 3 (unit equal to a sixteenth note). Note that an optimal cover is the minimum cardinality set of substrings that covers* s*). The final referenced array segment,* $\langle 4,9,7,6,0\rangle$*, does not contain a substring equal to its corresponding set of substrings of* s *(shown at left) and so it does not have any output (indicated by "N/A").*

time-point ordering of twelve distinct time points, resulting in a one-dimensional string of $n \times 12$ time points. We select suitable time-point orderings based on how well they induce a beat, according to a heuristic we developed (Bemman and Meredith, forthcoming). This string of $n \times 12$ time points serves as the foundation on which the pitch content is generated through equal-note-value strings using the steps outlined in this article. Figure 10 shows our proposed procedure for automating the compositional process seen in Babbitt's later works.

For each of the $(n \times 12) - 1$ time-point intervals, we compute, from its corresponding PcOM, a pseudorandom equal-note-value string according to the probabilities of immediate repeats, other repeats, and new pitch classes (described in Table 1) and ensuring its maximum length is not exceeded (as described in Equation 2). If this equal-note-value string has any referenced array segments, we adopt a greedy approach in which we select the one that produces a minimum cardinality set cover (or as optimal a cover as possible, as shown in Table 2), retaining the appropriate rests and ties. On the other hand, if this particular equal-note-value string does not have any reference, we generate a new string containing the same number of pitch classes and try again to find a reference. Because references can be difficult to find for larger lengths, attempting this process more than once ensures that smaller lengths are not favored and that a variety of string lengths will occur in the music. If still no reference is found after five attempts, we choose a new length and a new string of pitch classes, repeating this entire process of attempting to find a referenced array segment.

Figure 10. Proposed procedure for modeling the compositional process of Babbitt's late-practice pieces based on the all-partition array, time-point system, and equal-note-value strings (ENVSs).



Generated Piece

In this section, we present a novel piece automatically generated using the procedure proposed in this article. This piece, shown in Appendix A, is a work for flute and string quartet in which each instrument contains an all-partition array of 34 ordered mosaics in four lynes. In the flute part, its all-partition array has four lynes or "voices" distributed across two registers from C4–B5. In the quartet, each instrument similarly contains four lynes distributed across two registers with the cello from C2 to B3, the viola from C3 to B4, violin II from G3 to F♯5, and violin I from C4 to B5. Both the flute and quartet use the same sequence of time points so that each change in dynamic level aligns for every instrument. This piece makes use of only the first 17 of its all-partition array's 34 PcOMs. Ordinarily in Babbitt's music, the pitch material exhausts all PcOMS in the underlying all-partition array, resulting in a characteristically long piece of music. The completion of pitch material in these works typically signals the end of the piece, often without the time points having used all the TpcOMs (Bernstein 2017). Similarly, our piece ends with the completion of its 17 PcOMs by the flute, without all TpcOMs being used. At about its midpoint (the ninth PcOM), the piece changes the value of d (shown in Equation 2) from a sixteenth to a thirty-second note. This change allows for equal-note-value strings to now contain shorter durations and when combined with the manually indicated return to tempo marking in measure 22, creates a feeling of quickness in the second half of the piece.

It is important to note that pitch, onset, duration, voice, dynamic level, and meter have all been automatically generated in our piece. Meters have been chosen by dividing the prefix sum of time-point intervals by the number of units to a beat, from left to right until this value lies between two and six beats, inclusive. The fractional meters that result are more common in Babbitt's earlier practice, although they do appear at times in his later works. To ensure that not all instruments play together at all times, we have randomly chosen for a given time-point interval whether or not an equal-note-value string will occur for each instrument. Similarly, we have randomly chosen whether or not simultaneities will occur in an equal-note-value string of the stringed instruments. Such simultaneities are constrained to not exceed two and must belong to either segments 1 and 2 or 3 and 4 in their PcOM. This ensures pitch distance between simultaneities is not exceedingly large and impossible to perform on the instruments. Finally, the spelling of pitches (whether a flat or sharp) has been arbitrarily fixed to avoid flat symbols, favoring sharps throughout.

Conclusion

In this article, we have proposed a procedure for automating the compositional process used in Babbitt's later works. This process includes his use of techniques such as the all-partition array, time-point system, and equal-note-value strings. As our generated piece has demonstrated, these techniques alone are sufficient for generating a number of musical parameters that appear on the musical surface, including pitch, onset, duration, voice, and dynamic level. Additional parameters, such as articulation, phrase markings, tempo indication, and form, have not been included in our procedure. We would not be surprised, however, if further analysis of his sketches were to reveal similar algorithmic techniques for determining these additional parameters.

Acknowledgments

We would like to thank Laura Piras (flute), Clifton Harrison (violin I), Abigail Dance (violin II), Chris Brody (viola), and Anna Tam (cello) for their performance in the premiere of the piece presented in this article, which was held 28 September 2016 at the Vortex Jazz Club in London. The work reported here was carried out as part of the project "Learning to Create" (Lrn2Cre8), and we acknowledge the financial support of the Future and Emerging Technologies (FET) program within the seventh framework program for research of the European Commission, under FET grant number 610859.

References

Babbitt, M. 1955. "Some Aspects of Twelve-Tone Composition." *The Score and IMA Magazine* 12:53–61.

Babbitt, M. 1960. "Twelve-Tone Invariants as Compositional Determinants." *Musical Quarterly* 46(2):246–259.

Babbitt, M. 1961. "Set Structure as a Compositional Determinant." *Journal of Music Theory* 5(1):72–94.

Babbitt, M. 1962. "Twelve-Tone Rhythmic Structure and the Electronic Medium." *Perspectives of New Music* 1(1):49–79.

Babbitt, M. 1973. "Since Schoenberg." *Perspectives of New Music* 12(1/2):3–28.

Bemman, B., and D. Meredith. 2015. "Exact Cover Problem in Milton Babbitt's All-Partition Array." In T. Collins, D. Meredith, and A. Volk, eds. *Mathematics and Computation in Music*, *Lecture Notes in Artificial Intelligence*, volume 9110. Berlin: Springer, pp. 237–242.

Bemman, B., and D. Meredith. 2016. "Generating Milton Babbitt's All-Partition Arrays." *Journal of New Music Research* 45(2):184–204.

Bemman, B., and D. Meredith. forthcoming. "Predicting Babbitt's Orderings of Time-Point Classes from Array Aggregate Partitions in *None but the Lonely Flute* and *Around the Horn*." *Journal of Mathematics and Music*.

Bernstein, Z. 2017. "The Problem of Completeness in Milton Babbitt's Music and Thought." *Music Theory Spectrum* 38(2):241–264.

Cormen, T. H., et al. 2009. *Introduction to Algorithms*. Cambridge, Massachusetts: MIT Press, 3rd edition.

Ebcioğlu, K. 1987. "Report on the CHORAL Project: An Expert System for Harmonizing Four-Part Chorales." Technical Report RC12628. IBM T. J. Watson Research Center. Available online at global-supercomputing.com/people/kemal.ebcioglu/pdf/RC12628.pdf. Accessed 7 November 2017.

Leong, D., and E. McNutt. 2005. "Virtuosity in Babbitt's *Lonely Flute*." *Music Theory Online* 11(1):Article 4.

Mead, A. 1987. "About *About Time*'s Time: A Survey of Milton Babbitt's Recent Rhythmic Practice." *Perspectives of New Music* 25(1–2):182–235.

Mead, A. 1994. *An Introduction to the Music of Milton Babbitt*. Princeton, New Jersey: Princeton University Press.

Tanaka, T., B. Bemman, and D. Meredith. 2016a. "Constraint Programming Approach to the Problem of Generating Milton Babbitt's All-Partition Arrays." In M. Rueher, ed. *Principles and Practice of Constraint Programming*. Berlin: Springer, pp. 802–810.

Tanaka, T., B. Bemman, and D. Meredith. 2016b. "Integer Programming Formulation of the Problem of Generating Milton Babbitt's All-Partition Arrays." In *Proceedings of the International Conference on Music Information Retrieval*, pp. 171–177.

Appendix

On the following pages we present the complete score of a composition for flute and string quartet automatically generated in the style of Babbitt using the procedure proposed in this article.

The Computer as Specialist

(Who cares if you compose?)

Computer


Slower


16
20
♩ = 62
23


39
41
43