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Isomorphic Controllers and Dynamic Tuning: Invariant Fingering over a Tuning Continuum

In the Western musical tradition, two pitches are generally considered the “same” if they have nearly equal fundamental frequencies. Likewise, two pitches are in the “same” pitch class if the frequency of one is a power-of-two multiple of the other. Two intervals are the “same” (in one sense, at least) if they are an equal number of cents wide, even if their constituent pitches are different. Two melodies are the “same” if their sequences of intervals, in rhythm, are identical, even if they are in different keys. Many other examples of this kind of “sameness” exist.

It can be useful to “gloss over” obvious differences if meaningful similarities can be found. This article introduces the idea of *tuning invariance*, by which relationships among the intervals of a given scale remain the “same” over a range of tunings. This requires that the frequency differences between intervals that are considered the “same” are “glossed over” to expose underlying similarities. This article shows how tuning invariance can be a musically useful property by enabling (among other things) *dynamic tuning*, that is, real-time changes to the tuning of all sounded notes as a tuning variable changes along a smooth continuum. On a keyboard that is (1) tuning invariant and (2) equipped with a device capable of controlling one or more continuous parameters (such as a slider or joystick), one can perform novel real-time polyphonic musical effects such as tuning bends and temperament modulations—and even new chord progressions—

all within the time-honored framework of tonality. Such novel musical effects are discussed briefly in the section on dynamic tuning, but the bulk of this article deals with the mathematical and perceptual abstractions that are their prerequisite.

How can one identify those note layouts that are tuning invariant? What does it mean for a given interval to be the “same” across a range of tunings? How is such a “range of tunings” to be defined for a given temperament? The following sections answer these questions in a concrete way by examining two ways of organizing the perception of intervals (the rational and the ordinal), by defining useful methods of mapping an underlying just intonation (JI) template to a simple tuning system and scalic structure, and by describing the isomorphic mapping of that tuning system to a keyboard layout so that the resulting system is capable of both transpositional and tuning invariance.

Background

On the standard piano-style keyboard, intervals and chords often have different shapes in different keys. For example, the geometric pattern of the major third C–E is different from the geometric pattern of the major third D–F-sharp. Similarly, the major scale is fingered differently in each of the twelve keys. (In this article, the term “fingering” is used to denote the geometric pattern, without regard to which digits of the hand press which keys.) Other playing surfaces, such as the keyboards of Bosanquet (1877) and Wicki (1896) have the property that

each interval, chord, and scale type have the same geometric shape in every key. Such keyboards are said to be *transpositionally invariant* (Keislar 1987).

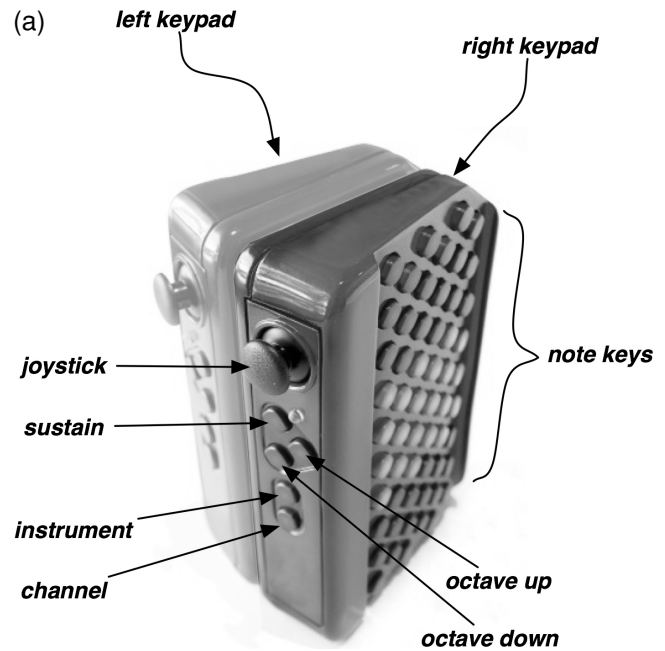
There are many possible ways to tune musical intervals and scales, and the introduction of computer and software synthesizers makes it possible to realize any sound in any tuning (Carlos 1987). Typically, however, keyboard controllers are designed primarily for the familiar 12-tone equal temperament (12-TET), which divides the octave into twelve logarithmically equal pieces. Is it possible to create a keyboard surface that is capable of supporting many possible tunings? Is it possible to do so in a way that analogous musical intervals are fingered the same throughout the various tunings, so that (for example) the 12-TET fifth is fingered the same as the just fifth and the 17-TET fifth? (Just intervals are those consisting of notes whose constituent frequencies are related by ratios of small integers; for example, the just fifth is given by the ratio 3:2, and the just major third is 5:4.)

This article answers this question by presenting examples of two related *tuning continua* (parameterized families of tunings where each specific tuning corresponds to a particular value of the parameter) that exhibit *tuning invariance* (where, on an appropriate instrument, all intervals and chords within a specified set have the same geometric shape in all of the tunings of the continuum). A keyboard that is transpositionally invariant, tuning-invariant, and has a continuous controller has three advantages. First, having a single set of fingerings within and across all keys of any given tuning makes it easier to visualize the underlying structure of the music. Second, having this same single set of interval shapes across the tuning continuum makes it easier for musicians to explore the use of alternative tunings such as the various meantones, Pythagorean, 17-TET, and beyond. Third, assigning the continuous parameter to a control interface enables a unique form of expression, for example, dynamically tuning (or retuning) all sounded notes in real time, where the scalic function of the notes remains the same, even as the tuning changes.

In this article, the Wicki layout is used to concretely demonstrate the formation of the pitches and notes on a practical keyboard surface, though

Figure 1. Thumtronics' forthcoming USB-MIDI controller, the Thummer, contains two Wicki-layout keyboards, each with 57 note-controlling buttons. A variety of controllers in-

clude two thumb-operated joysticks and optional internal motion sensors. The Thummer can be held like a concertina (a) or laid flat, like most keyboard instruments (b).



other keyboard layouts such as those of Fokker (1955) or Bosanquet could have been used instead. The Wicki layout can be conveniently mapped to a standard computer keyboard, facilitating the exploration of the ideas presented in this article. Thumtronics' forthcoming Thummer music controller (see www.thummer.com), shown in Figure 1, uses the Wicki note layout by default.

There are several technical, musical, and perceptual questions that must be addressed to realize a keyboard that is both transpositionally and tuning invariant. First, there must be a range of tunings over which pitch intervals—and therefore their fingerings—remain in some sense the “same.” This requires that differently tuned intervals be identified as serving the same role; for instance, the 12-TET fifth must be identified with the just fifth and the 19-TET fifth. Said differently, tuning invariance requires that there be a number of distinguishable intervals by which the invariance can be measured, because to say that two numerically different intervals are both “perfect fifths,” it is necessary to identify a perfect fifth as an interval distinguishable from a major third, or a perfect fourth, or a diminished fifth, and so forth. This issue of the identity of



musical intervals is discussed in detail in the next section (“Intervals”) by contrasting rational and ordinal modes of interval identification. The rational mode is determined by the correspondence of an interval to a low-ratio JI interval, and the ordinal mode is determined by the number of scale notes an interval spans.

Second, there must be a tuning system that is itself transpositionally invariant with regard to both forms of identification. This requires that each and every note in the system has identical intervals above and below, and that the presumed temperament-mapping of JI to this tuning system is consistent. Such a tuning system is called a *regular tuning system*, and the embodiment of a such a temperament-mapping in a regular tuning system is called a *regular temperament*. These are defined more fully in the section “Tuning Systems and Temperaments.”

Third, given such a tuning system, it is necessary to define useful sets of scales. The section entitled “Scales” focuses on those scales known as MOS or well-formed, which have a number of musically advantageous properties.

Fourth, it is necessary to layout-map the regular temperament to a keyboard or button-field in a manner that maintains transpositional and tuning invariance. The layout-mappings described in the section “Button-Lattices and Layouts” translate the generating intervals of the tuning to the keyboard surface. It is shown that transpositional invariance is identical to linearity of the layout-mapping. Successive sections then provide examples of keyboard layouts that are (and others

that are not) invariant in both transposition and tuning. The tuning continuum pictured in Figure 2 provides the primary example of this article. It begins at 7-TET, and by varying the size of the perfect fifth, it moves continuously through various meantone tunings, 12-TET, Pythagorean, and many other tunings, ending at 5-TET, while retaining fingering invariance throughout. Finally, a musical example illustrates static snapshots of the dynamic retuning process.

Intervals

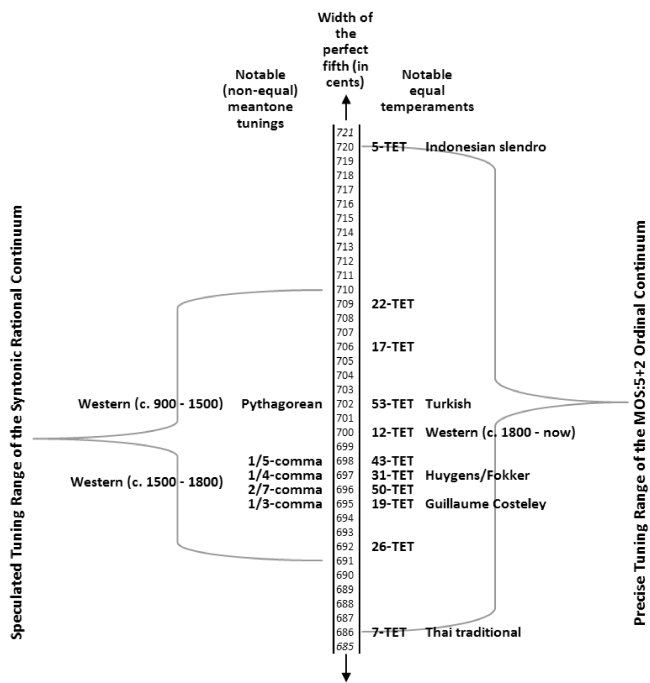
This section investigates how intervals are identified and distinguished, and it discusses criteria by which two numerically different intervals may be said to play analogous roles in different tuning systems. Intervals may be identified—and therefore discriminated—in at least two ways, here defined as the rational and the ordinal.

Rational Identification

The *rational* mode of identification is presumed to occur primarily for harmonic intervals, namely, those formed from simultaneously sounded notes. It presumes that just intervals that are perceived as consonant act as perceptual and cognitive landmarks (a template) against which sounded intervals can be mentally compared and identified. This is reasonably uncontroversial for orchestral instruments

Figure 2. The syntonic rational continuum and MOS:5+2 ordinal continuum (both defined later). The figure also shows the historical and ethnic usages of many notable tunings found within these

continua (compiled from information in Barbour 1951, Lorentz 2001, Sethares 2004, and the Huygens-Fokker Foundation Web site at www.xs4all.nl/huygensf).



with harmonic spectra. For inharmonic, computer-generated sounds, inharmonic bells, and non-Western instruments such as the metallophones of the Indonesian gamelan, these sensory consonances may occur at different intervals (Sethares 2004), and so other templates may be more appropriate.

For sounds with harmonic spectra, when a sounded interval is tuned close to a small-integer-ratio JI interval (such as $3:2$ or $5:4$), it may be heard as a representation of that ratio. Using semiotic terminology, the sounded interval is an indexical signifier of the just ratio it approximates. For example, when an interval is tuned to 702 cents (the closest integer value to $3:2$), it is likely to be heard as a representation of the $3:2$ just fifth. As the interval's tuning is moved away from 702 cents, it gradually moves to a state where it is likely to be heard as an imperfect representation of $3:2$. (It will sound more or less "out of tune.") As the tuning is moved still further from 702 cents, the perceived interval will eventually no longer represent $3:2$ (and not even an out-of-tune $3:2$). At this point, the just interval is no longer signified by the sounded interval. When an interval

is not just, but is within its range of rational identification, it is called a *tempered* interval.

Ordinal Identification

The *ordinal* mode of identification is presumed to occur primarily for melodic intervals (formed from successively sounded notes). It presumes that when an interval is played as part of an aesthetically consistent or conventionalized scale, it is identified by the number of scale notes (or steps) it spans. For example, Wilson (1975) writes that "our perception of Fourth-ness is not just acoustic, i.e., $4/3$ -determined, it is melodic and/or rhythmic-influenced to a high degree" (p. 1). Like rational identification, this is an indexical signification but one that requires the presence of a scalic background (i.e., context) to give meaning to "second-ness," "third-ness," "fourth-ness," etc. For a scale to serve as a background, it must have intrinsic aesthetic consistency (i.e., be perceived as somehow complete and "correct") and/or be conventionalized (i.e., made familiar through repetitive use).

The diatonic scale can serve as a scalic background for common-practice music. For example, against the background of a C-major (diatonic) scale, the intervals C–D, D–E, E–F, F–G, and so on, are heard as "seconds" because they each span two adjacent scale notes; the intervals C–E, D–F, E–G, F–A, and so forth, are heard as "thirds" because they each span three successive scale notes.

Dual Identification

The two modes of identification overlap: the rational mode plays a part in the identification of melodic intervals, the ordinal in the identification of harmonic intervals. For example, a melodic interval of approximately $3:2$ will usually be heard as "in tune" or "out of tune" according to its proximity to this just interval; similarly, a harmonic interval approximating $3:2$ will, in a traditional diatonic context, be heard as spanning five notes of the scale.

Furthermore, in real-world music it is not always

possible to make a strict distinction between harmonic intervals and melodic intervals. An arpeggio is at least partially a harmonic structure. The bass note of a “stride bass” pattern, which is sounded only for the first and third beats of a bar, cognitively grounds the rest of the bar. Counterpoint, which is the interweaving of many melodies into a coherent harmonic structure, blurs the distinction between melody and harmony.

Thus the rational and ordinal modes of identification are intertwined. Indeed, in Western tonal music, there is a consistent linkage between the number of steps an interval contains and its harmonic ratio. For example, the interval that spans three scale notes (i.e., the third) is commonly an interval that is close to 5:4 or 6:5 (hence the names major third and minor third). Similarly, the interval that spans four scale notes (i.e., the fourth) is typically an interval that is close to 4:3. Furthermore, where these links differ, the interval often has a tonally dissonant function that requires resolution to a more stable interval. (For example, an augmented second commonly resolves to a perfect fourth; a diminished fifth commonly resolves to a major third.)

In conventionalized musical systems such as Western tonal music, where particular step sizes and harmonic ratios are consistently associated, ordinality can (by association) symbolically signify ratio, and ratio can (by association) symbolically signify ordinality. This means that, within such a conventionalized context, the tuning ratio over which an interval can still signify a given just ratio may be wider than expected if it were judged by analyzing harmonies isolated from the musical context. In common practice, for example, the conventional association of “5:4-ness” and “third-ness” means that a melodic third can be tuned very wide but still signify 5:4. For example, the ultra-sharp supra-Pythagorean major thirds (greater than 408 cents) that are sometimes used by string players for expressive intonation (Sundberg et al. 1989) may be harmonically uncomfortable, but they still signify the same musical interval as the just quarter-comma meantone thirds of 386 cents. (They are, after all, different expressions of the same notated interval.) Thus the tuning range over which a given

interval can be identified and therefore discriminated from other intervals has “fuzzy” boundaries that are context-dependent.

Tuning Ranges of Invariant Identification

The previous discussion suggests that it may be advantageous to define the tuning range over which an interval preserves both its rational and ordinal identity as that interval’s *tuning range of invariant identification*. Because there are two modes by which an interval can be identified, there are two relevant tuning ranges to be considered that themselves depend on a listener’s innate abilities, experiences, and training. Musical context is also important; the tuning range of invariant rational identity may also change based on the spectrum and/or timbre of the sounds (Sethares 2004). For these reasons, the final judgment as to the specific tuning boundaries may best lie in the hands of the artist and not the theorist.

Nonetheless, in the presence of a scalic background, invariant ordinal identification can be precisely bounded; furthermore, given a scale in a regular tuning system (as defined subsequently), the size of every interval in the scale is determined by the values of the generating intervals. This implies that the range of generator tunings over which all of that scale’s intervals can be retuned but still maintain their ordinal identity—that scale’s *ordinal continuum*—is limited by the points at which one or more of that scale’s steps shrink in size to zero (or equivalently, the points at which two of the scale steps “cross”).

In a regular temperament (as defined in the next section), the sizes of all the intervals that are capable of being rationally identified are controlled by the values of its generating intervals. This implies that the range over which all of that temperament’s identifiable intervals can be retuned but still maintain their rational identity—that temperament’s *rational continuum*—is delimited by context and subject-dependent “fuzzy” boundaries.

Figure 2 shows the precisely bounded continuum for the twelve-note chromatic scale generated by

fifths and octaves. Also shown on this chart is a conjectured “fuzzy” boundary for the rational continuum of the regular temperament defined by the syntonic comma (which is explained in the next section). This conjectured range has been estimated by assuming that only the common practice consonances are rationally identifiable, and that their rational identification switches from one common practice consonance to another at the tuning that is midway between their just tunings. In common-practice music, these two continua are conventionally associated, though the ordinal continuum is wider than the rational.

Tuning Systems and Temperaments

A *tuning system* is defined here to be a collection of precisely tuned musical intervals. There are many ways in which the intervals may be chosen: A “boutique” tuning system might have all of its intervals chosen arbitrarily, and another tuning system might be generated by a predefined mathematical procedure (McLaren 1991). The *regular tunings* form one class of tuning systems in which all of the intervals are generated multiplicatively from a finite number of *generating intervals* (or *generators*). Such tuning systems ensure that every given note has the same set of intervals above and below it as every other note in the system; this means that regular tuning systems are inherently transpositionally invariant. An example regular tuning system is 3-limit JI (also known as Pythagorean tuning), which has two generators $G_1 = 2$ and $G_2 = 3$, and consists of all products of the form $G_1^i G_2^j = 2^i \times 3^j$, where i and j are integers. Thus the intervals of 3-limit JI can all be found in a series of stacked just perfect fifths, allowing for octave equivalence. In general, a regular tuning is characterized by n generators G_1 to G_n and consists of all intervals $G_1^{i_1} G_2^{i_2} \dots G_n^{i_n}$, where the i_1, i_2, \dots, i_n are integer-valued exponents.

Altering the tuning of a generator affects the tuning of the system in a predictable way. For example, the perfect fifth in 3-limit JI is $G_1^{-1} G_2$ (i.e., $2^{-1} \times 3$). If a (non-JI) regular tuning is created by changing the value of G_2 , the value of the fifth, and all other intervals, changes in a patterned way. Assigning the

magnitude of one or more of the generating intervals to a control interface provides a convenient means to “navigate” the tuning continuum. Particular values within this continuum may produce some intervals that approximate JI intervals and so are rationally identifiable; we might consider, therefore, that there has been a mapping of JI intervals to that tuning system.

For such a temperament-mapping to be transpositionally invariant, it must be linear, though it need not be invertible (i.e., it need not be one-to-one). The embodiment of such a temperament-mapping in a suitable tuning system is called a *regular temperament*, and it can be characterized by the small JI intervals called *commas* that are tempered to unison (Smith 2006). This means that a regular temperament is characterized by its temperament-mapping, not its tuning, so any given temperament has a range of suitable tunings. To be concrete, two or more intervals a_1, a_2, \dots, a_n are said to be *multiplicatively dependent* if there are integers z_1, z_2, \dots, z_n , not all zero, such that $a_1^{z_1} a_2^{z_2} \dots a_n^{z_n} = 1$. If there are no such z_i , then the a_i are said to be *multiplicatively independent*. The *rank* of a tuning system is the number of multiplicatively independent intervals needed to generate it. A regular temperament typically has lower rank than the JI system that is temperament-mapped to it (i.e., the mapping is non-invertible). When the temperament-mapping loses rank, all intervals can no longer be just. However, as long as there is a range of generator values over which the intervals are correctly rationally identified, the regular temperament can be considered to be valid.

This is analogous to the way a projection of the three-dimensional surface of a globe to a two-dimensional map inevitably distorts distance, area, and angle. However, so long as the countries have identifiable shapes, the projection can be considered valid. Different map projections result in different distortions, and some map projections are more or less suitable to specific purposes. Some projections (such as the Mercator Projection) have the virtue of wide familiarity; so it is also with temperament-mappings (such as those that lead to 12-TET).

For example, 3-limit JI can be temperament-mapped to a one-dimensional system using an ap-

appropriate comma $G_1^a G_2^b = 1$, where a and b are integers. One notable tempering retains the octave $G_1 = 2$ and tempers G_2 to $2^{19/12} \approx 3$, which requires that $a = 19$ and $b = -12$, resulting in the familiar 12-TET. This comma can be interpreted musically by saying that in this temperament, 19 octaves minus twelve equal-tempered perfect twelfths equals a unison. A second example is in 5-limit JI, which consists of all intervals of the form $2^i 3^j 5^k$, where i , j , and k are integers. This can be reduced to a two-dimensional regular temperament by choosing G_1 (typically near 2), G_2 (near 3), and G_3 (near 5) so that

$$G_1^a G_2^b G_3^c = 1 \quad (1)$$

where a , b , and c are specified integers. Equation 1 defines a comma that is tempered to unison. The well known *syntonic* comma (which has an untempered tuning of $81/80$) is the special case where $a = -4$, $b = 4$, and $c = -1$; tempering the generators so that Equation 1 holds, produces various tunings of the *syntonic temperament* (as illustrated in Figure 2). The comma can be solved for one of its terms as $G_3 = G_1^{-a/c} G_2^{-b/c}$, and a typical interval of the regular temperament can be written in terms of two generating intervals $\alpha = G_1^{\text{GCD}(a,c)/|c|}$ and $\beta = G_2^{\text{GCD}(b,c)/|c|}$ where $\text{GCD}(n, m)$ is the greatest common divisor of n and m . Thus $G_1 = \alpha^{c/|\text{GCD}(a,c)|}$, $G_2 = \beta^{c/|\text{GCD}(b,c)|}$, $G_3 = \alpha^{-a\text{SIGN}(c)/|\text{GCD}(a,c)|} \beta^{-b\text{SIGN}(c)/|\text{GCD}(b,c)|}$, and a typical interval $G_1^i G_2^j G_3^k$ of the JI is temperament-mapped to $\alpha^{(ci-ak)/|\text{GCD}(a,c)|} \beta^{(cj-bk)/|\text{GCD}(b,c)|}$. In matrix notation, the vector $(i, j, k)^T$ is temperament-mapped by

$$\mathbf{R} = \begin{bmatrix} \frac{|c|}{\text{GCD}(a,c)} & 0 & \frac{-a \cdot \text{SIGN}(c)}{\text{GCD}(a,c)} \\ 0 & \frac{|c|}{\text{GCD}(b,c)} & \frac{-b \cdot \text{SIGN}(c)}{\text{GCD}(b,c)} \end{bmatrix} \quad (2)$$

to

$$\begin{bmatrix} \frac{i|c| - ak\text{SIGN}(c)}{\text{GCD}(a,c)} & \frac{j|c| - bk\text{SIGN}(c)}{\text{GCD}(b,c)} \end{bmatrix}^T$$

where T is the transpose operator. Thus, the temperament-mapping is linear and (typically) loses rank. For the syntonic comma,

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 4 \end{bmatrix} \quad (3)$$

and any note $(i, j, k)^T$ is mapped to $(i - 4k, j + 4k)^T$. For example, a just major third $5/4 = 2^{-2} 3^0 5^1$ is represented by the vector $(-2, 0, 1)^T$. This is temperament-mapped by G to $(-6, 4)^T$, which represents the tempered interval $\alpha^{-6}\beta^4$. If α were tempered to 2 (no change in the octave) and β were tempered to $2^{30/19}$ (the 19-TET twelfth), then $\alpha^{-6}\beta^4 = 2^{-6 \cdot 2^{120/19}} \approx 1.24469 \approx 5/4$ is the 19-TET approximation to the just major third. Similarly, if α were tempered to 2 and β were tempered to $3 \times (81/80)^{-1/4}$ (the quarter-comma meantone fifth), then $\alpha^{-6}\beta^4 = 5/4$ is the justly tuned major third found in the quarter-comma meantone tuning.

Syntonic Rational Continuum

The *rational continuum* of a rank- r regular temperament is here defined as that (r -dimensional) range of generator tunings within which the rational identity of all rationally identifiable intervals is maintained. The rational continuum of the syntonic temperament is therefore called the *syntonic rational continuum*. As described previously, the tuning range of this continuum is context-dependent and not easily specified a priori.

The syntonic rational continuum extends beyond the narrower range implied by “meantone,” which usually refers to the range of syntonic tunings providing reasonably pure-sounding tunings (for sounds with harmonic spectra) and/or which have an established historical use—a range of approximately 19-TET to 12-TET. (See Figure 2.)

Scales

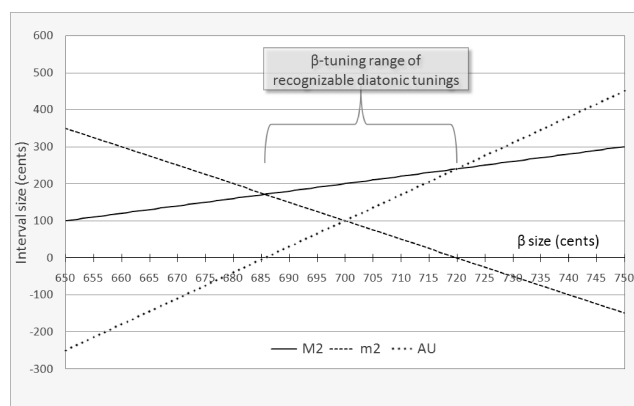
A *scale* is here defined to be a subset of a tuning system used for a specific musical purpose. When using a rank-2 tuning system with generators α and β , a scale can be simply constructed by stacking integer powers of β and then reducing (dividing or multiplying by integer powers of α) so that every term lies between 1 and α . This is called an α -reduced β -chain, and it produces a scale that repeats at intervals of α ; $\alpha = 2$, representing repetition at the octave, is the most common value. Any arbitrary segment of an

α -reduced β -chain can be used to form a scale, and the number of notes it contains is called its *cardinality*. For a given tuning of α and β , α -reduced β -chains with certain cardinalities are called *moment of symmetry (MOS) scales* (Wilson 1975; also known as *well-formed scales* after Carey and Clampitt 1989). Such scales have a number of musically advantageous properties: they are distributionally even, which means that the scale has two step sizes that are distributed as evenly as possible (Clough et al. 1999); they have constant structure, which means that every given interval always spans the same number of notes (Grady 1999). Two familiar MOS scales generated by $\alpha = 2$ and $\beta \approx 3/2$ are the five-note pentatonic scale (e.g., D, E, G, A, C) and the seven-note diatonic scale (e.g., C, D, E, F, G, A, B); other β -values generate MOS scales with quite different intervallic structures. These provide fertile resources for non-standard microtonal scales.

Every MOS scale with specified generators and cardinality has a valid tuning range over which it can exist; beyond this range, notes must be added to or removed from the scale to regain distributional evenness and constant structure. The boundaries of this tuning range mark the tuning points at which some of the scale's steps shrink to unison, and therefore the tuning points at which the ordinal identity of intervals in that scale changes. An MOS scale generated by α and β with cardinality c has a two-dimensional valid tuning range of $\alpha^{p/q} < \beta < \alpha^{r/s}$, where p/q and r/s are adjacent members of a Farey sequence of order $c-1$. (A *Farey sequence* of order n is the set of irreducible fractions between 0 and 1 with denominators less than n , arranged in increasing order.)

For example, the 12-note MOS scale generated by $\alpha = 2$ and $\beta \approx 3/2$, which is the chromatic scale that provides the background for common practice music, has a valid tuning range of $2^{4/7} < \beta < 2^{3/5}$ (4/7 and 3/5 being adjacent members of the Farey sequence of order 11). Expressing this range in simpler terms, the "perfect fifth" is between $1200 \times 4/7 = 685.7$ and $1200 \times 3/5 = 720$ cents in size. Consider the familiar seven-note diatonic scale as the tuning traverses this range: starting with a generating interval β tuned to the 12-TET fifth $2^{7/12}$, the scale consists of five large steps (whole tones) and two small steps

Figure 3. Sizes of the major second (M2), minor second (m2), and augmented unison (AU), over a range of β generator tunings.



(half tones). As β is increased, the whole tones grow larger and the half tones grow smaller. When β reaches $2^{3/5}$, the semitones disappear completely; above $2^{3/5}$, there is no seven-note MOS scale available, because the seven-note scale contains three different step sizes. On the other hand, as β is decreased below $2^{7/12}$, the tones get smaller and the semitones get larger. When β reaches $2^{4/7}$, the tones and semitones become the same size; below $2^{4/7}$, the MOS scale's internal structure changes so that it has five small intervals and two large intervals, a scale structure that is the inverse of the familiar diatonic scale. These changing step sizes are illustrated in Figure 3. This tuning range is equivalent to Blackwood's "range of recognizable diatonic tunings" (1985). The valid tuning ranges for MOS scales are, therefore, a generalization of the concept of range of recognizability beyond the familiar diatonic/chromatic context.

The MOS scale that is recognizably diatonic is called MOS:5+2 (i.e., it contains five large steps and two small steps). The valid tuning range of this MOS scale is $2^{4/7} < \beta < 2^{3/5}$ (see Figure 2) and it preserves, therefore, the ordinal identity of the diatonic and chromatic intervals used in common practice.

When intervals are played as a part of a conventionalized or aesthetically consistent scale, they can be ordinally identified by the number of notes or intervals they span. Owing to their distributional evenness and constant structure, MOS scales are likely to be heard as intrinsically aesthetically consistent, so the valid tuning range of any given MOS scale is equivalent to the range within which all of

its intervals are ordinally invariant. Furthermore, these ranges hold for many musically useful alterations of MOS scales such as the harmonic minor scale (A, B, C, D, E, F, G-sharp).

MOS:5+2 Ordinal Continuum

The *ordinal continuum* of an MOS: $a+b$ scale, which has a large steps and b small steps, is here defined as the (two-dimensional) range of generator tunings that gives this scale form. It is equivalent to $\alpha^{p/q} < \beta < \alpha^{r/s}$, where p/q and r/s are adjacent members of a Farey sequence of order $2a + b - 1$. The ordinal continuum of MOS:5+2 is, therefore, called the *MOS:5+2 ordinal continuum*.

Button Lattices and Layouts

Instruments capable of playing a number of discrete pitches may use many buttons (or keys). For an instrument to have transpositional invariance, it is necessary that the buttons are arranged in a regular and spatially repeating pattern; such a structure is called a *lattice* (Insall, Rowland, and Weisstein 2007). If the lattice has one dimension, it may be called a *button row*; if it has two dimensions, it may be called a *button field*. A *layout* is here defined as the physical embodiment of an invertible, but not necessarily linear, layout-mapping from a regular temperament to an integer-valued button lattice. In the same way that a regular temperament has a finite number of generating intervals (e.g., α and β) that generate all its intervals, a lattice can be represented with a finite number of *basis vectors* that generate all its vectors. The logical means to layout-map from a temperament to a button lattice is to map the temperament's generating intervals to the lattice's basis vectors.

Let $L : \mathbf{Z}^n \rightarrow \mathbf{Z}^n$ map from the n generating intervals of the temperament to the n -dimensional button lattice. For example, with $n = 2$, the temperament contains two generating intervals α and β . Any interval of the temperament can be expressed as a two-vector (j, k) representing the interval $\alpha^j \beta^k$. The standard basis for the temperament consists of the

generating intervals $(1, 0)$ and $(0, 1)$, and the layout-mapping L is the 2×2 matrix

$$L = \begin{bmatrix} \psi_x & \omega_x \\ \psi_y & \omega_y \end{bmatrix} \quad (4)$$

which transforms the temperament's generating intervals into the lattice's basis vectors $\boldsymbol{\psi} = (\psi_x, \psi_y)$ and $\boldsymbol{\omega} = (\omega_x, \omega_y)$. The elements of L must be integers (or else some intervals will be layout-mapped to locations without buttons), L must be invertible (that is, the determinant of L is nonzero, or else either some buttons would have no assigned note or some notes would have no corresponding button), and the determinant of L must be ± 1 (or else the inverse will not be integer-valued). This mapping provides the mathematical setting for two results. First, if the layout-mapping L is linear, the layout is transpositionally invariant (Theorem 1). Second, the converse holds as well: if the keyboard layout is transpositionally invariant, the layout-mapping L must be linear (Theorem 2).

Theorem 2 justifies the use of linear (matrix) notation for the layout-mapping. Proof of the theorems is given in Appendices A and B. The theorems presume an infinitely sized button-lattice; physical keyboards will necessarily have finite size, and the linearity will be violated at the edges. The finite size also limits the number of octaves that can be realized. There are many possible layout matrices L , and several concrete examples are given in the next section. The choice of L impacts the ease with which particular scales and chords can be fingered, as well as the balance between the number of octave-reduced intervals and overall octave range that can fit on a keyboard of a given geometry. For some layouts, such as that of Fokker, the playability-related metrics vary greatly as the tuning is changed, whereas these metrics are more stable across a wide range of tunings on some other layouts, such as the Wicki. Identifying and quantifying such metrics is an important area for future investigation.

Because a layout-mapping is invertible, by definition, it can be linear only if it does not lose rank. Linear and invertible mappings are called *isomorphic*, and henceforth all layouts with this type of layout-mapping will be referred to as *isomorphic layouts*. The theorems show that given a regular

temperament, transpositional invariance requires an isomorphic layout; and given a regular temperament and an MOS-scalic background, tuning invariance requires that the tunings of the generating intervals (as embodied in the layout) remain within that temperament's and MOS scale's tuning range for invariant identification. This means that transpositional invariance on a button-row is only possible for a rank-one (i.e., equal) temperament. To get transpositional invariance across the tuning range of rank-two temperaments (like the non-equal meantone tunings), a button field of at least two dimensions is required. Because it is difficult to conceive of a button lattice operating effectively in more than two dimensions, only rank-one and rank-two temperaments are considered in this article. The following examples illustrate these points by demonstrating concrete isomorphic layouts that are invariant in both transposition and tuning.

Examples of Layout-Mappings

The examples in this section use the syntonic temperament tuned to quarter-comma meantone, 12-TET, and 19-TET, and assume an MOS:5+2 scalic background—design choices that are compatible with common-practice tonal music. Two types of layouts are shown. One-dimensional button rows provide the simplest setting; drawings such as Figures 4–6 should be interpreted as consisting of a single row of identical buttons. For two-dimensional button fields, there are several ways in which the buttons can be arranged: in a rectangular grid, in offset rows (like brickwork), in a hexagonal grid, or as a tiling of parallelograms. The Wicki layout provides our primary two-dimensional example, though obviously other layouts could be used.

Layout-Mapping from a Rank-One Temperament to a Button Row

The familiar 12-TET has a single generating interval α of $2^{1/12}$ (100 cents) and tempers out the syntonic comma (among other commas). An octave consists of twelve of these generating intervals, a perfect

fifth contains seven, and a major third contains four. The layout-mapping can be either $L = 1$ or $L = -1$. If the 12-TET generating interval is layout-mapped to $+1$, the notes progress sequentially higher in pitch from left to right, which corresponds to a linear keyboard as shown in Figure 4.

The tuning range of rational invariance for this temperament is small. Increasing the size of the generator by just two cents increases the size of the octave by 24 cents. Though the piano is frequently tuned with stretched octaves, this stretching is typically less than about half a cent on the generating interval (about six cents per octave). The tuning range of ordinal invariance is unbounded, because no amount of retuning of the generator changes the order of the notes. For this case, the ordinal range is not a useful measure.

Owing to the isomorphic layout-mapping, this layout has the property that any rationally identifiable interval or chord (such as the major triad) is fingered the same for any position on the button-row. Starting on any note (for instance C), a major triad is played using the buttons four and seven steps to the right. The geometric shape of a major triad is, therefore, always 0–4–7 for all transpositions. Similarly, a melody is fingered the same wherever it is located on the button-row. Starting on any note (for instance C), the melody Re–Mi–Fa–Fi–Sol is played using a button, then a button two steps to the right, then one more step to the right, then one more step to the right. The geometric shape of this melody is, therefore, always 0–2–3–4–5 for all transpositions.

A different rank-one temperament that also tempers out the syntonic comma (among others) is 19-TET, as shown in Figure 5. This temperament has a single generating interval $\alpha = 2^{1/19}$ (approximately 63.2 cents), and the octave consists of 19 of these generating intervals. The perfect fifth is eleven generating intervals wide, and the major third is six generating intervals wide. Any isomorphic layout-mapping of 19-TET (e.g., $L = 1$) has, like the above 12-TET example, a small range of rational tuning invariance and a similar transpositional invariance. However, the fingering of both the major triad 0–6–11 and the Re–Mi–Fa–Fi–Sol melody 0–3–5–6–8 are

Figure 4. The one-dimensional keyboard layout $L = 1$ in 12-TET.

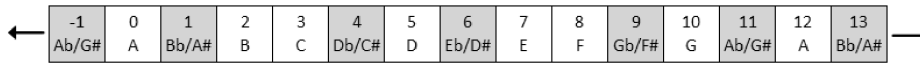


Figure 4

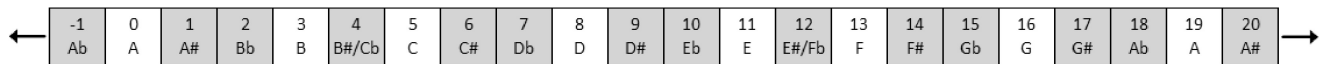


Figure 5

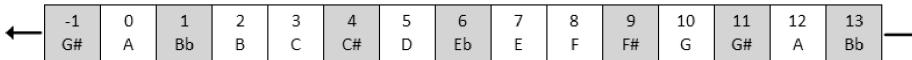


Figure 6

different from the 12-TET design in Figure 4. This shows concretely how the linear one-dimensional design fails to be tuning invariant.

Layout-Mapping from a Rank-Two Temperament to a Button Row

A layout-mapping is invertible by definition, so a layout-mapping from a rank-two regular temperament to a one-dimensional button-row must be nonlinear. As established previously, a nonlinear (non-isomorphic) layout-mapping “breaks” transpositional invariance. A common example of a nonlinear layout-mapping is a piano-style layout, which takes a finite subset of the notes produced by a rank-two meantone (such as quarter-comma), and layout-maps them in pitch order to a twelve-note-per-octave keyboard, as illustrated in Figure 6. The physical/geometrical irregularity of the piano keyboard, interpreted as a row of black keys interspersed with a row of white keys, is collapsed here to a single row for the sake of simplicity.

Such layout-mappings (as applied to standard, geometrically irregular keyboards) were used in the 17th century (Barbour 1951), though sometimes A-flat was used in place of G-sharp. In quarter-comma (or any other meantone tuning requiring two generating intervals), major thirds (e.g., C–E) are tuned differently than diminished fourths (e.g., C-sharp–

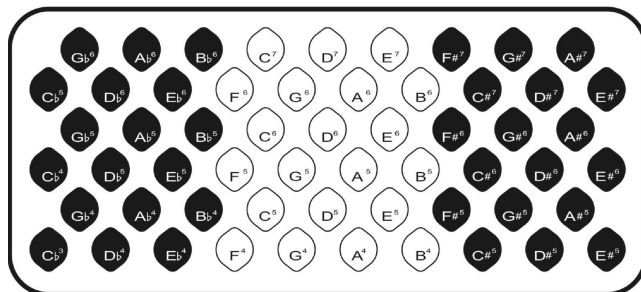
Figure 6. Nonlinear piano-style layout for a rank-2 temperament, such as quarter-comma meantone.

F). In the nonlinear layout of Figure 6, ascending four steps sometimes produces a major third and sometimes produces a diminished fourth, thus breaking transpositional invariance. For example, starting from C and proceeding four steps to E produces a just major third of size 386 cents, which is clearly identifiable as 5:4. On the other hand, starting at C-sharp and ascending four steps to F produces a diminished fourth of 427 cents. This is a so-called “wolf” interval that is unlikely, in this context (where there exist more closely tuned major thirds), to be rationally identified as 5:4.

The only way to gain transpositional invariance for this layout is to tune the fifths to $2^{7/12}$, which is the only (octave-reduced) tuning that does not differentiate between major thirds and diminished fourths. This is consistent with the theorem because when the fifth is tuned to $2^{7/12}$ and the octave is tuned to 2, the two generating intervals are no longer multiplicatively independent. In this case, the rank of the temperament has collapsed to one and the tuning has become 12-TET. A similar argument shows that for a button-row with N notes per octave, any tuning that produces an n -TET where $n = N$ will have transpositional invariance, but for this tuning only.

To summarize: On a button-row with keys of fixed size, the physical size of any given interval is different in 12-TET, 17-TET, 19-TET, quarter-

Figure 7. The position of notes using the Wicki layout on a Thummer keyboard.



comma, or any alternative tuning. In each tuning, the performer's fingers have to press different keys to produce intervals that are the same (within the definition of a given temperament). A keyboard that is tuning-invariant is preferable, so that the performer's fingers press the same keys to produce the same intervals (within a given temperament), independent of the current tuning.

Layout-Mapping from a Rank-Two Temperament to a Button Field

One solution to these problems is to use a two-dimensional button-field so that the layout-mapping from the rank-two syntonic temperament to the button-field is isomorphic. The simplicity afforded by the invariant fingering of an isomorphic layout is illustrated in the following examples using the Wicki layout.

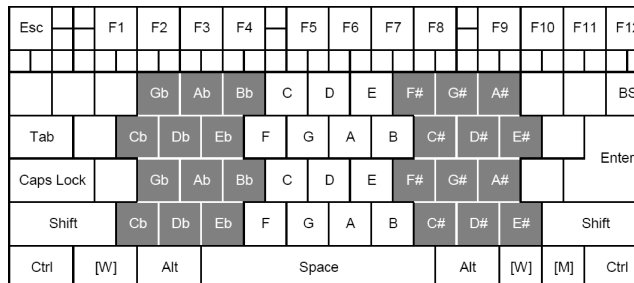
To be concrete, Wicki is a layout-mapping to a hexagonal button-field (which can be approximated by rotating an integer valued button-field by 45°), such that the two generating intervals α and β are layout-mapped to the basis vectors $\psi = (1, 1)^T$ and $\omega = (1, 0)^T$, i.e.,

$$L = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

as illustrated in Figure 7.

This layout provides a good balance of octave-reduced intervals versus overall octave range for relatively small button fields (such as the QWERTY computer keyboard as illustrated in Figure 8) when using syntonic tunings ($\alpha \approx 3/2$), and it also functions well over many non-syntonic tuning continua (where β may take any value). Additionally, the

Figure 8. The position of notes using the Wicki layout on a QWERTY keyboard.



Wicki layout allows a wide range of pitches to be played with a single hand, and all notes of any MOS scale are always clustered in a simple vertical band. (Proof of the latter property is beyond the scope of this article.) However, it has a less obvious relationship between pitch and button position than is found, for instance, in the design of Bosanquet. (For syntonic tunings, Bosanquet has an "easterly" pitch axis, like the standard keyboard, while the Wicki layout has a less obvious "north-northeasterly" pitch axis.) The basic principles illustrated in the following examples may also be applied to other regular temperaments, MOS scales, and layouts.

Tuning Ranges of Invariance

Assuming that the generating intervals are $\alpha = 2$ (an octave), and $\beta = F$ (an alterable tempered perfect fifth), the pitch order of buttons on a Wicki layout that is ten buttons wide has seven different configurations as the tuning of F moves across the MOS:5+2 ordinal continuum of $2^{4/7} < F < 2^{3/5}$. Figure 9 shows the pitch order of the buttons, starting from D3 (numbered 0) up to D4 (an octave above), as F traverses this range. Despite these different configurations, every melodic interval that is "legal" to common practice has its order preserved. (Intervals such as G-sharp-A-flat, F-flat-E-sharp, or C-flat-B-sharp typically have no melodic function in common-practice music.) For example, observe that although the pitch order changes as the tuning is raised from $2^{4/7}$ to $2^{3/5}$, the diatonic notes (colored white in this figure) remain in the same order, and so do the "legal" chromatic intervals. The shape of the chromatic melody Re-Mi-Fa-Fi-Sol is indicated by crosses,

Figure 9. Pitch order of notes across the MOS:5+2 ordinal continuum's tuning range. The notes within the D3 to D4 octave have been highlighted. (a) $F =$

$2^{4/7}$ (7-TET); (b) $2^{4/7} < F < 2^{7/12}$; (c) $F = 2^{7/12}$ (12-TET); (d) $2^{7/12} < F < 2^{10/17}$; (e) $F = 2^{10/17}$ (17-TET); (f) $2^{10/17} < F < 2^{3/5}$; (g) $F = 2^{3/5}$ (5-TET).

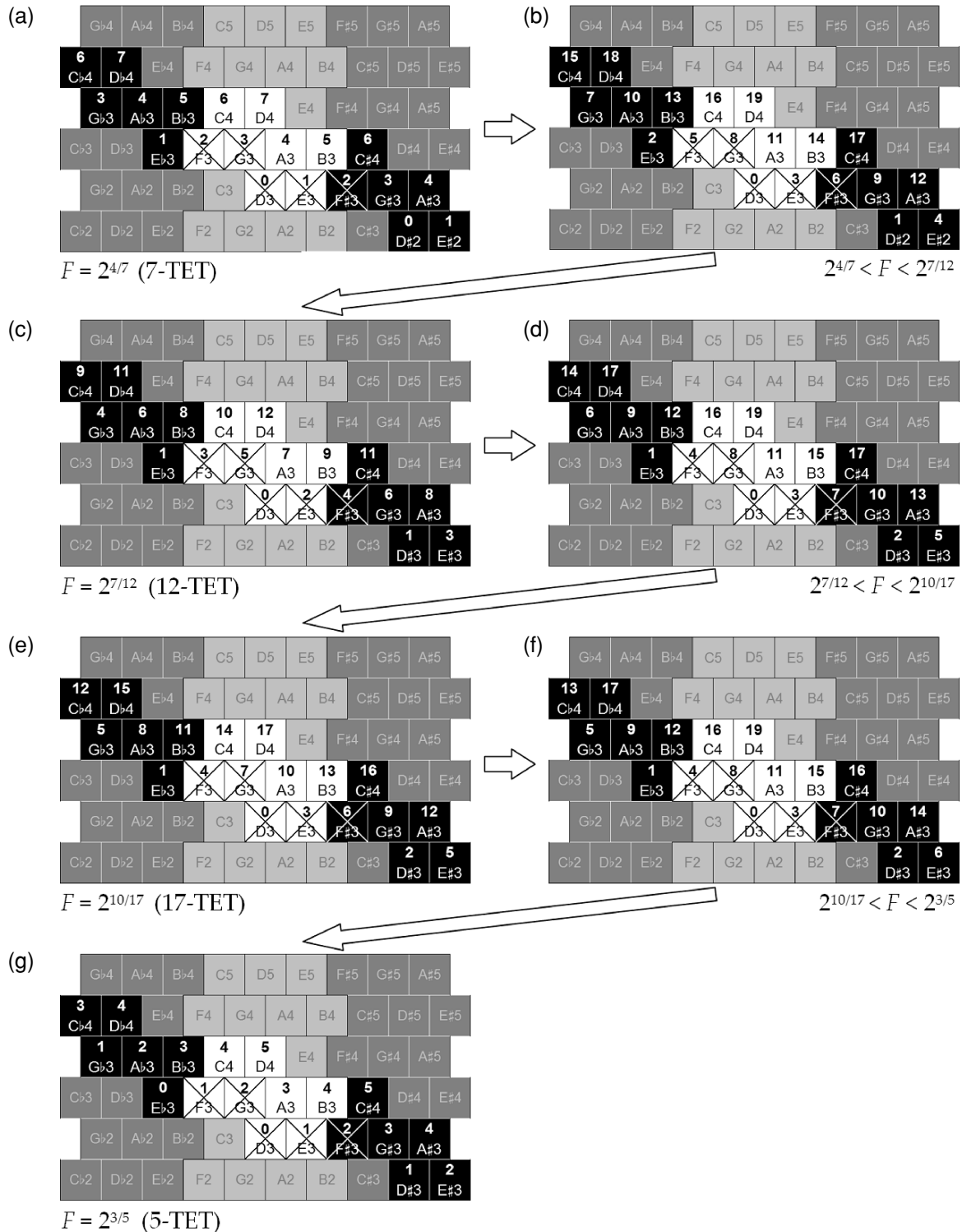
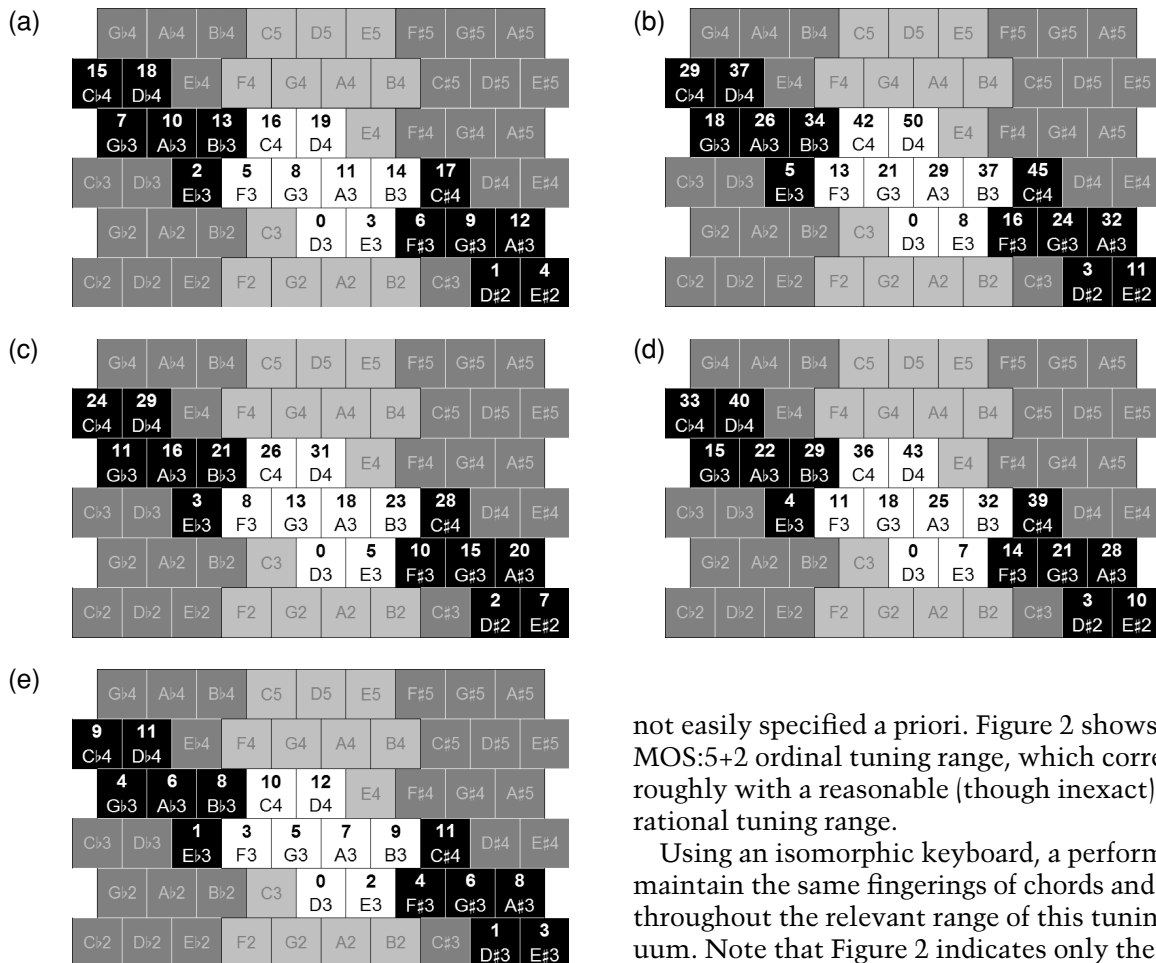


Figure 10. The step positions for a selection of meantone n -TETs. The notes within the D3 to D4 octave have been highlighted. For n -TETs where $n > 19$, some of the notes fall beyond the edges of

this button-field. (a) Meantone n -TET (19-TET); (b) meantone n -TET (50-TET); (c) meantone n -TET (31-TET); (d) meantone n -TET (43-TET); (e) meantone n -TET (12-TET).



and the pitch order within this shape remains the same, only breaking down when $F \leq 2^{4/7}$ or $F \geq 2^{3/5}$.

Another natural consequence of invariant fingering is that the steps of the various n -TETs falling within any given continuum automatically line up to the correct button position. This is illustrated by Figure 10, which shows a selection of meantone n -TETs.

Figure 11 shows the positions on the Wicki layout of the common practice harmonic consonances that can be signified over a range of values of the fifth F . This example is again centered on the note D3 (an arbitrary choice).

As discussed previously, the limits for invariant rational identification are context-dependent and

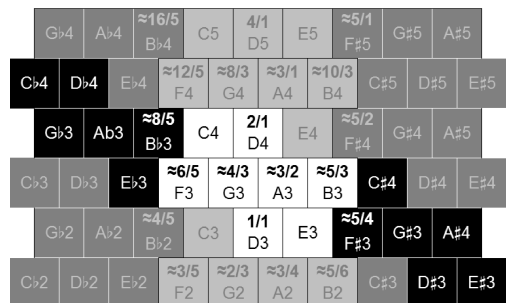
not easily specified a priori. Figure 2 shows the MOS:5+2 ordinal tuning range, which corresponds roughly with a reasonable (though inexact) syntonic rational tuning range.

Using an isomorphic keyboard, a performer can maintain the same fingerings of chords and melodies throughout the relevant range of this tuning continuum. Note that Figure 2 indicates only the syntonic temperament, but a given tuning can belong to more than one regular temperament. For example, it is likely that 53-TET tuning falls within the tuning ranges of invariant rational identity for both the syntonic and the schismatic temperament (which tempers the 5-limit generators of Equation 1 with $a = -15$, $b = 8$, and $c = 1$).

Dynamic Tuning

Keyboard designs that are invariant in both transposition and tuning offer several performance possibilities to the computer-based musician and composer. All the tunings of the syntonic (or MOS:5+2) continuum can be characterized by the width of their tempered fifth. The exact value of this fifth can be

Figure 11. Position of the signified common practice harmonic consonances using the Wicki layout and the syntonic temperament.



linked to a single slider, modulation wheel, joystick axis, or other control device. As the performer moves this control, thereby changing the width of the tempered perfect fifth, the pitches of all sounded notes will change correspondingly to match the current tuning. If the extremes of the control's range correspond to the extremes of the MOS:5+2 ordinal continuum, then sliding that control from one end to the other will result in the continuum of tunings shown in Figures 2 and 9.

For example, one might choose to perform a simple I–IV–V–I pattern such as shown in Figure 12. The three staves show the score as it would be written in conventional notation; the numerical annotations show the scale steps in each of three tunings (31-TET, 12-TET, and 17-TET) as the parameter β moves from $2^{18/31}$ to $2^{7/12}$ to $2^{10/17}$. As shown in Figure 9, the fingering remains the same throughout the continuum. Thus, players of a tuning-invariant keyboard may be able to transfer competence in one tuning (such as 12-TET) to others; hard-won manual dexterity can be transferred directly to other tunings within the continuum. This is not possible on the piano keyboard, guitar fretboard, or other common musical interfaces.

The tuning can also be changed dynamically during performance. By analogy with pitch bend, this might be called “tuning bend,” where the exact tuning of each note in each interval is changed in response to the physical motion of the controller. For example, the performer might push up into a supra-Pythagorean tuning (like 17-TET) to give melodies more expressive power, drop down to 12-TET to perform a smooth enharmonic modulation, and then drop down to a quasi-meantone tuning (like 31-TET) for more pleasing triads. This may give

Figure 12. A simple chord progression annotated with the scale-step numbers in 31-TET, 12-TET, and 17-TET. On an isomorphic keyboard, this chord progression can be played with identical fingerings in all keys and all tunings of the syntonic rational continuum.



keyboard players greater flexibility in mimicking the kinds of expressive pitch deviations that string (and aerophone) players reveal when altering their intonation phrase by phrase (Sundberg et al. 1989). Moreover, tuning bends are not limited to monophonic implementations. Rocking the controller back-and-forth describes a kind of (polyphonic) vibrato where each note might have a different amount and direction of pitch deviation.

Discussion

Music education, performance, and composition can benefit from the transpositional and tuning invariance provided by isomorphic button fields. The piano-style keyboard is incapable of achieving the advantages of an isomorphic keyboard for three reasons. First, it is essentially one-dimensional, and

one-dimensional layouts can only have transpositional invariance for equal temperaments. Second, its offset pattern of twelve keys per octave is only well suited to tunings that use twelve (or fewer) notes per octave. Third, the piano keyboard's offset white and black keys hide the consistent patterns of 12-TET—let alone any other tunings—forcing students to learn each key's scales, chords, and other properties by rote.

In contrast, with an isomorphic button-field, a student need only learn the geometric shape of a given interval once (within a given temperament), and thereafter apply that knowledge to all occurrences of that interval, independent of its location within a key, across keys, or across tunings. This reduces rote memorization considerably and engages the student's visual and tactile senses in discerning the consistency of music's patterns.

Furthermore, the MOS:5+2 ordinal continuum includes the tunings of many cultures and eras. At one of the continuum's extremes, for example, 7-TET provides a close approximation to traditional Thai music (Morton 1980) and to traditionally tuned African balafon music (Jessup 1983), and at the other extreme, 5-TET is close to Indonesian slendro (Surjodiningrat, Sudarjana, and Susanto 1993). Between these two extremes can be found many tunings that have been explored in European history, such as quarter-comma meantone, Pythagorean tuning, and today's ubiquitous 12-TET. A student, having mastered the use of an isomorphic button field in any one of these tunings, can apply the same motor skills and music theory to all the rest, making multi-cultural music education considerably easier.

Although this article has focused on the syntonic temperament and diatonic and chromatic scales, the benefits of tuning invariance are gained by all rank-two regular temperaments, such as schismatic, porcupine, magic, and hanson (Tonalsoft 2007), and their respective MOS scales. The different geometries of different temperaments and MOS scales on the same keyboard make it easier for students to compare, contrast, and understand them, experiencing those differences through sight and touch in addition to hearing.

Performers can also benefit from isomorphic key-

boards. Consider that accompanists are often asked to transpose a piece at sight to accompany the range of a particular vocalist. Being able to transpose on the fly may be the mark of a master musician today, but with an isomorphic keyboard, the task becomes significantly easier. With an isomorphic keyboard and dynamic tuning, performers can also execute real-time polyphonic tuning bends, temperament modulations, and other effects that are simply impossible on a piano keyboard or guitar fretboard.

Composers can also benefit from the transpositional and tuning invariance of isomorphic keyboards. Just as Beethoven was the first to exploit the novel technology of the metronome to specify precise tempi (Stadlen 1967), composers writing for isomorphic keyboards can indicate appropriate tunings for pieces that are meant to imply a baroque (quarter-comma), medieval Ars Nova (Pythagorean), or modern atonal (12-TET) feel.

Of course, there are also disadvantages to such isomorphic keyboards compared to that of the piano. The playing of diatonic intervals connected by parallel motion is more complex than on the white keys of the piano, and with some layouts, there is a less obvious linear relationship between pitch height and location on the keyboard. Although isomorphic keyboards have been proposed and built in the past, they do not have a large installed user base or a deep repertoire. This paper has shown that isomorphic keyboards can be coupled with appropriate computer technology to provide the deeper benefits of tuning invariance and dynamic tuning.

References

- Barbour, J. M. 1951. *Tuning and Temperament: A Historical Survey*. East Lansing, Michigan: Michigan State College Press. Reprint, New York: Da Capo Press, 1973. Reprint, New York, Dover, 2004.
- Blackwood, E. 1985. *The Structure of Recognizable Diatonic Tunings*. Princeton, New Jersey: Princeton University Press.
- Bosanquet, R. H. M. 1877. *Elementary Treatise on Musical Intervals and Temperament*. London: Macmillan. Cited in Helmholtz (1877).
- Carey, N., and D. Clampitt. 1989. "Aspects of Well-Formed Scales." *Music Theory Spectrum* 11:187–206.

- Carlos, W. 1987. "Tuning: At the Crossroads." *Computer Music Journal* 11(1):29–43.
- Clough, J., N. Engebretsen, and J. Kochavi. 1999. "Scales, Sets, and Interval Cycles." *Music Theory Spectrum* 10:19–42.
- Fokker, A. D. 1955. "Equal Temperament and the Thirty-One-Keyed Organ." *The Scientific Monthly* 81(4):161–166. Available online at www.xs4all.nl/~huygensf/doc/fokkerorg.html. (Accessed 26 March 2007.)
- Grady, K. 1999. "CS." Message posted to *Alternate Tunings Mailing List*. Available online at launch.groups.yahoo.com/group/tuning/message/5244. (Accessed 11 October 2006.)
- Helmholtz, H. L. F. 1877. *On the Sensations of Tone*. Trans. A.J. Ellis. Reprint, New York: Dover, 1954.
- Insall, M., T. Rowland, and E. Weisstein. 2007. "Point Lattice." *MathWorld—A Wolfram Web Resource*. Available online at mathworld.wolfram.com/PointLattice.html. (Accessed 4 June 2007.)
- Jessup, L. 1983. *The Mandinka Balafon: An Introduction with Notation for Teaching*. La Mesa, California: Xylo Publications.
- Keislar, D. 1987. "History and Principles of Microtonal Keyboards." *Computer Music Journal*, 11(1):18–28. Published with corrections in 1988 as "History and Principles of Microtonal Keyboard Design," Report STAN-M-45, Stanford University. Available online at ccrma.stanford.edu/STANM/stanms/stanm45. (Accessed 13 June 2007.)
- Lorentz, R. 2001. "19-TET in a Renaissance Chanson by Guillaume Costeley." Paper presented at Microfest, Claremont, California, April 6.
- McLaren, B. 1991. "General Methods for Generating Musical Scales." *Xenharmonikon* 13:20–44.
- Morton, D. 1980. "The Music of Thailand." In E. May, ed. *Music of Many Cultures*. Berkeley, California: University of California Press, pp.63–82.
- Sethares, W. A. 2004. *Tuning, Timbre, Spectrum, Scale*, 2nd ed. London: Springer-Verlag.
- Smith, G.W. 2006. "Xenharmony." Available online at bahamas.eshockhost.com/xenharmony. (Accessed October 2006.)
- Stadlen, P. 1967. "Beethoven and the Metronome." *Music and Letters* 48:330–349.
- Sundberg, J., F. Anders, and L. Frydén. 1989. "Rules for Automated Performance of Ensemble Music." *Contemporary Music Review* 3(1):89–109.
- Surjodiningrat, W., P. J. Sudarjana, and A. Susanto. 1993. *Tone Measurements of Outstanding Javanese Gamelan in Yogyakarta and Surakarta*. Yogyakarta, Indonesia: Gadjah Mada University Press.
- Tonalsoft. 2007. "Equal Temperament." *Encyclopedia of Microtonal Music Theory*. Available online at tonalsoft.com/enc/e/equal-temperament.aspx. (Accessed 26 March 2007.)
- Wicki, K. 1896. "Tastatur für Musikinstrumente." Swiss patent 13329.
- Wilson, E. 1975. "Letter to Chalmers Pertaining to Moments Of Symmetry/Tanabe Cycle." Available online at www.anaphoria.com/mos.PDF. (Accessed 26 March 2007.)

Appendix A: Proof of Theorem 1

Let $f: \mathbf{Z}^2 \rightarrow \mathbf{Z}^2$ be an invertible mapping from the two-dimensional generators of a rank-2 temperament to the two-dimensional button-lattice. Any note N in the temperament can be represented in terms of its generators α and β as $N = \alpha^n \beta^m$, which can be written as the vector (n, m) for $n, m \in \mathbf{Z}$. The note is located on the button-lattice at $f(N) = f(n, m)$. An interval I is a ratio of two notes

$$\frac{N_1}{N_2} = \frac{\alpha^{n_1} \beta^{m_1}}{\alpha^{n_2} \beta^{m_2}} \quad (\text{A.1})$$

which can be written (j, k) , where $j = n_1 - n_2$ and $k = m_1 - m_2$. A layout is *transpositionally invariant* if every fixed interval I is fingered in the same manner, i.e., if

$$f(N_1) - f(N_2) = f(N_3) - f(N_4) \quad (\text{A.2})$$

whenever

$$\frac{N_1}{N_2} = \frac{N_3}{N_4} = I \quad (\text{A.3})$$

Thus, transpositional invariance requires that the difference in locations between notes on the keyboard depends only on the interval (the j and k) and not on the particular notes (the n_1 and m_1). Expanding Equation A.2 yields

$$f\left(\begin{matrix} n_1 \\ m_1 \end{matrix}\right) - f\left(\begin{matrix} n_1 + j \\ m_1 + k \end{matrix}\right) = f\left(\begin{matrix} n_3 \\ m_3 \end{matrix}\right) - f\left(\begin{matrix} n_3 + j \\ m_3 + k \end{matrix}\right) \quad (\text{A.4})$$

where we used the equality of the intervals from Equation A.3, i.e., that $n_2 = n_1 + j$, $m_2 = m_1 + k$, $n_4 = n_3 + j$, and $m_4 = m_3 + k$. If the mapping f is linear, then this can be rewritten

$$f\begin{pmatrix} n_1 \\ m_1 \end{pmatrix} - f\begin{pmatrix} n_1 \\ m_1 \end{pmatrix} + f\begin{pmatrix} j \\ k \end{pmatrix} = f\begin{pmatrix} n_3 \\ m_3 \end{pmatrix} - f\begin{pmatrix} n_3 \\ m_3 \end{pmatrix} + f\begin{pmatrix} j \\ k \end{pmatrix} \quad (\text{A.5})$$

which collapses to an identity for any n_i and m_i . This demonstrates that linear layout mappings are transpositionally invariant. An analogous argument works for any invertible rank- r mapping $f: \mathbf{Z}^r \rightarrow \mathbf{Z}^r$.

Appendix B: Proof of Theorem 2

The converse of Theorem 1 is demonstrated by showing that transpositional invariance implies linearity of f . This will first be shown in the scalar case, where the defining Equation (A.1) for transpositional invariance is that for all fixed intervals i ,

$$f(x + i) - f(x) = f(y + i) - f(y) \quad (\text{B.1})$$

for every $x, y \in \mathbf{Z}$. We also assume that $f(0) = 0$. Linearity of f is shown by demonstrating additivity and homogeneity.

Additivity

Because Equation B.1 holds for all x and y , it must hold for $x = j$ and $y = 0$. Substituting into Equation B.1 shows $f(j + i) - f(j) = f(0 + i) - f(0)$. Because $f(0) = 0$, this can be rearranged to show $f(i + j) = f(i) + f(j)$.

Homogeneity

Since Equation B.1 holds for all x and y , it must hold for $x = i$ and $y = 0$. Substituting into Equation B.1 shows $f(i + i) - f(i) = f(0 + i) - f(0)$. Because $f(0) = 0$, this can be rearranged to show $f(2i) = 2f(i)$. Induction can be used to show the general case. Suppose that $f((k - 1)i) = (k - 1)f(i)$. Let $x = (k - 1)i$ and $y = 0$. Substituting into Equation B.1 shows $f((k - 1)i + i) - f((k - 1)i) = f(i)$. Using the inductive hypothesis, this can be rearranged to show $f(ki) = (k - 1)f(i) + f(i) = kf(i)$. The generalization to two (or n) dimensions is straightforward.